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Editor : Anil Ahlawat

editorial

Only Sugar-coated Pills are popular

Many good students leave the mathematics and physics stream to choose anything that is offered as an alternative paper. This is unfortunate because they miss a very good source of joy just because the packaging was not attractive. The way to combat this syndrome is to arrange in every college, university and schools, a series to popular lectures on not only mathematics but every other subject including the history of mathematics.

It is unfortunate that when we talk to students, many in the high school stage itself are ready to experiment with any other subject if only they can avoid mathematics. This fear continues even when they become professors. On the other side of the fence, there are the majority of professors in the physical science stream who try to avoid anything connected with botany and physiology. The cure is the same – exposure to lectures by great professors and doctors. If professors and lecturers in art subjects such as English and other languages do not like the courses such as mathematics, the dislike is genuine. It is even fear rather than dislike. What is the solution?

First arrange extension lectures by the top popular mathematics professors and research scientists where there is no marking system. Follow up by explaining the mathematics used in modern research. Knowledge of statistics, calculus, group theory and quantum mechanics are absolutely necessary for any research professor or student. These mathematics only help us to draw the correct conclusions. If teaching physics itself needs sugar-coating, teaching mathematics needs as double coating of sugar and chocolate! The lesson given by homeopathy doctors is this – even good medicines should be given as sugar pills.

Anil Ahlawat
Editor

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 149

JEE MAIN

1. The sum of the first 2015 terms of the sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1, 2, 2, ... is

(a) 3966 (b) 3967 (c) 3968 (d) 3969

2. The graph of a function $y = f(x)$ is symmetric about the line $x = a$ and $x = b < a$. The function is periodic with period

(a) $a + b$ (b) $a - b$ (c) $2(a + b)$ (d) $2(a - b)$

3. The sides of a triangle ABC are 5, 6, 7. $P(x, y)$ is a point in the plane of the triangle such that $PA^2 + PB^2 + PC^2 = 70$. The locus of P is a circle of radius

(a) $\frac{8}{3}$ (b) $\frac{10}{3}$ (c) $\frac{11}{3}$ (d) $\frac{13}{3}$

4. $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e + e\dfrac{x}{2}}{ex^2} =$

(a) $\frac{11}{24}$ (b) $\frac{13}{24}$ (c) $\frac{7}{12}$ (d) $\frac{5}{8}$

JEE ADVANCED

5. A straight line L through the point A (1, 2) meets the line $x + y = 4$ at the point B. If $AB = \sqrt{2}$, the slope of L is

(a) $2+\sqrt{3}$ (b) $2-\sqrt{3}$

(c) $-2+\sqrt{3}$ (d) $-2-\sqrt{3}$

6. A straight line L is a tangent to the parabola $y^2 = 4x$ and a normal to the parabola $x^2 = \sqrt{2}\,y$. The distance of the origin from L is

(a) 0 (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{3}{\sqrt{5}}$ (d) 2

COMPREHENSION

A and B are two points on the parabola $y^2 = 4ax$ such that the normals at A and B meet at the point $C(at^2, 2at)$ on the parabola

7. The locus of the orthocentre of triangle ABC is a parabola of latus rectum

(a) $\frac{a}{2}$ (b) a (c) $\frac{3a}{2}$ (d) $2a$

8. The locus of the circumcentre of triangle ABC is parabola of latus rectum

(a) $\frac{a}{2}$ (b) a (c) $\frac{3a}{2}$ (d) $2a$

INTEGER MATCH

9. A straight line with negative slope cuts x-axis at A and y-axis at B. O is the origin. If it passes through the point $\left(1, \frac{1}{2}\right)$, the minimum value of the perimeter of the triangle OAB is

MATCHING LIST

10. In a triangle ABC, $r_1 = 21$, $r_2 = 24$, $r_3 = 28$

	Column-I		Column-II
P.	The sum of the digits of s =	1.	3
Q.	The sum of the digits of a =	2.	6
R.	The sum of the digits of b =	3.	8
S.	The sum of the digits of c =	4.	10

	P	Q	R	S
(a)	1	2	3	4
(b)	2	3	4	1
(c)	3	4	1	2
(d)	4	1	2	3

See Solution set of Maths Musing 148 on page no. 69

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

JEE WORKOUTS

PAPER-I

SECTION-I

(Multiple Correct Answer Type)

This section contains multiple correct answer(s) type questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

1. Let $f(x) = \ln |x|$ and $g(x) = \sin x$. If A is the range of $f(g(x))$ and B is the range of $g(f(x))$, then

(a) $A \cup B = (-\infty, 1]$ (b) $A \cup B = (-\infty, \infty)$

(c) $A \cap B = [-1, 0]$ (d) $A \cap B = [0, 1]$

2. $\lim\limits_{x\to 0} \dfrac{8}{\sin^8 x}\left(1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)=$

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{32}$ (c) $\dfrac{1}{64}$ (d) $\dfrac{1}{8}$

3. If $f(x)=\begin{cases}3x^2+12x-1, & -1\le x\le 2\\ 37-x, & 2<x\le 3\end{cases}$, Then

(a) $f(x)$ is increasing in $[-1,2]$

(b) $f(x)$ is continuous in $[-1,3]$

(c) $f'(2)$ does not exist

(d) $f(x)$ has maximum at $x = 2$

4. If $x\in\left[0,\dfrac{\pi}{2}\right]$, then $\sin x + 2x$

(a) $=\dfrac{3x}{\pi}(x+1)$ (b) $\le\dfrac{3x}{\pi}(x+1)$

(c) $\ge\dfrac{3x}{\pi}(x+1)$ (d) none of these

5. The angle between the pair of lines

$(x^2+y^2)\left(\dfrac{\cos^2\theta}{4}+\sin^2\theta\right)=\left(\dfrac{x}{\sqrt{3}}-y\sin\theta\right)^2$ is

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$

(c) $\dfrac{\pi}{3}$ (d) $2\,\theta$.

6. Tangents are drawn from a point P on the circle $C : x^2 + y^2 = a^2$ to the circle $C_1 : x^2 + y^2 = b^2$. If these tangents cut the circle C at Q and R and if QR is a tangent to the circle C_1, then the area of triangle PQR is

(a) $3\sqrt{3}\,b^2$ (b) $\dfrac{3\sqrt{3}}{4}a^2$

(c) $\dfrac{3\sqrt{3}}{2}ab$ (d) $2\sqrt{3}\,ab$

7. If the plane $x + y + z = 1$ is rotated through 90° about its line of intersection with the plane $x - 2y + 3z = 0$, the new position of the plane is

(a) $x - 5y + 4z = 1$ (b) $x - 5y + 4z = -1$

(c) $x - 8y + 7z = 2$ (d) $x - 8y + 7z = -2$.

8. The general solution of the equation

$3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$ is

(a) $n\pi$ (b) $2n\pi$

(c) $2n\pi+\dfrac{\pi}{2}$ (d) $2n\pi+\dfrac{\pi}{4}$

9. There exists a triangle ABC satisfying the conditions

(a) $b\sin A<a, A>\dfrac{\pi}{2}$ (b) $b\sin A>a, A>\dfrac{\pi}{2}$

(c) $b\sin A>a, A<\dfrac{\pi}{2}$ (d) $b\sin A<a, A<\dfrac{\pi}{2}, b>a$

10. If $z\neq 0$, then the maximum value of $|z^i|$ is

(a) $|z|$ (b) 1

(c) $e^{-\pi}$ (d) e^{π}

SECTION-II

(Integer Answer Type Questions)

This section contains questions. The answer to each of the question is a single digit integer, ranging from 0 to 9.

11. Let z be a complex number such that $|z| = 1$. The number of complex numbers z, such that z^3 is pure imaginary, is

12. Let $\vec{a}, \vec{b}, \vec{c}$ be coplanar unit vectors such that $\vec{b}\cdot\vec{c} = \cos\alpha$, $\vec{c}\cdot\vec{a} = \cos\beta$, $\vec{a}\cdot\vec{b} = \cos\gamma$. Then the value of $\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma$ is

13. In a triangle ABC, if a, b, c are in A.P., then the value of $\frac{ca}{rR}$ is

14. The number of values of x in $[-\pi, \pi]$ such that $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, is

15. If $I = \int_0^{\pi/2} \frac{x}{\sin x}dx$ and $J = \int_0^1 \frac{\tan^{-1} x}{x}dx$, then $\frac{I}{J}$ is

16. If $\frac{dy}{dx} = xy + x^3y^3$ and $y(0) = 1$, then $3\left(y\left(\frac{1}{2}\right)\right)^2$ is

17. If the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$ is $\frac{m}{n}$, where m and n are relatively prime positive integer, then $m + n$ is

18. If $\lim\limits_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{ex^2} = \frac{m}{n}$, where m and n are relatively prime positive integers, then $3m - n$ is

19. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is such that the length of tangents to it from the points $(1, 0)$, $(2, 0)$ and $(3, 2)$ are respectively $1, \sqrt{7}$ and $\sqrt{2}$. Then $|c|$ is

20. Let $f(x) = \frac{ax+b}{cx+d}$, $x \neq -\frac{d}{c}$. If $f(5) = 5$, $f(13) = 13$ and $f(f(x)) = x$ for all x except $\frac{-d}{c}$, then the number which is not in the range of f is

PAPER-II

SECTION-I

(Single Correct Answer Type)

This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

21. If $f(x) = \int f(x)\,dx$, then

$\int\left[f''(x) + \{f'(x)\}^2 + \{f(x)\}^3\right]dx = 0$ will give

$\sum\limits_{r=1}^{3} \frac{\{f(x)\}^r}{r} =$

(a) $f''(x)$ (b) constant
(c) $f^3(x) + c$ (d) None of these

22. Let $f(x) = [x] - x$ (where $[\cdot]$ is *g. i.* function) and

$g(x) = \lim\limits_{n\to\infty} \frac{\{f(x)\}^{4n} - 1}{\{f(x)\}^{4n} + 1}$ then $\int \sum\limits_{r=1}^{2014} \{g(x)\}^r\,dx =$

(a) 2014 (b) 2015
(c) 0 (d) None of these

23. If α, β, γ be the angles made by a line with axes so that $2\left(\frac{\tan^2\alpha}{1+\tan^2\alpha} + \frac{\tan^2\beta}{1+\tan^2\beta} + \frac{\tan^2\gamma}{1+\tan^2\gamma}\right) = 3\sec^2\frac{\theta}{2}$, then $\theta =$

(a) $\pi/12$ (b) $\pi/10$ (c) $\pi/6$ (d) $\pi/3$

24. If $a, b, c, d > 0$; $x \in R$ and $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$, then $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$ is equal to

(a) 1 (b) -1 (c) 0 (d) 2

25. If A and B are square matrices of the same order, then which of the following is always true?

(a) $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$
(b) A and B are non-zero and $|AB| = 0 \Leftrightarrow |A| = 0$ and $|B| = 0$
(c) $(AB)^{-1} = A^{-1}B^{-1}$
(d) $(A + B)^{-1} = A^{-1} + B^{-1}$

26. If z lies on the circle $|z - 1| = 1$, then $\frac{z-2}{z}$ equals

(a) 0 (b) 2
(c) -1 (d) none of these

27. If A, B and C are three events such that $P(B) = \frac{3}{4}$, $P(A \cap B \cap C') = \frac{1}{3}$ and $P(A' \cap B \cap C') = \frac{1}{3}$, then $P(B \cap C)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{9}$

28. If $2f(xy) = (f(x))^y + (f(y))^x \ \forall\ x, y$ and $f(1) = 2$, then $\sum_{n=1}^{9} f(n) =$

(a) 1022 (b) 1023 (c) 1024 (d) 1025

29. The differential equation satisfied by all the circles with centres on the line $x = y$ is

(a) $(x - y)y_2 = (1+ y_1)(1 + y_1^2)$

(b) $(x - y)y_2 = (1 + y_1)(1 + y_1)^2$

(c) $(x - y)y_1^2 = (1 + y_1^2)$

(d) $(x - y)y_1^2 = (1 + y_1)y_2^2$

30. If $\cos (x - y)$, $\cos x$, $\cos (x + y)$ are in H.P., then $\cos x \cdot \sec \frac{y}{2}$ is

(a) $\pm\sqrt{2}$ (b) $\pm\sqrt{3}$

(c) $\sqrt{2}$ (d) None of these

SECTION-II

(Comprehension Type)

This section contains paragraphs. Based upon each paragraph, multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

Paragraph for Question Nos. 31 to 33

Let L_1 and L_2 be the lines $x + 2y - z - 3 = 0 = 3x - y + 2z - 1$ and $2x - 2y + 3z - 2 = 0 = x - y + z + 1$.

31. The angle between the lines is

(a) $\frac{\pi}{4}$ (b) $\cos^{-1}\frac{1}{3}$

(c) $\cos^{-1}\sqrt{\frac{2}{53}}$ (d) $\cos^{-1}\sqrt{\frac{2}{83}}$

32. The distance of the origin from the point of intersection of L_1 and L_2 is

(a) $\sqrt{22}$ (b) $\sqrt{33}$ (c) $2\sqrt{11}$ (d) $\sqrt{11}$

33. The distance of the origin from the plane through the lines is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{4\sqrt{2}}$

Paragraph for Question Nos. 34 to 36

Normals are drawn from the point (3, 2) to the parabola $y^2 = 4x$. The tangents at the feet of the normals form a triangle ABC. The distance of the focus of the parabola from the

34. Centroid is

(a) $\sqrt{2}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $2\sqrt{2}$

35. Orthocentre is

(a) $\sqrt{2}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $2\sqrt{2}$

36. Circumcentre is

(a) $\sqrt{2}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $2\sqrt{2}$

Paragraph for Question Nos. 37 to 40

Let A be $n \times n$ matrix with determinant $|A| \neq 0$. Then

37. $|\text{adj } A| =$

(a) $|A|$ (b) $|A|^n$ (c) $|A|^{n-1}$ (d) $|A|^{n+1}$

38. $|\text{adj } A|^{-1} =$

(a) A (b) $\frac{A}{|A|}$ (c) $\frac{A}{|A|^n}$ (d) $\frac{A}{|A|^{n-1}}$

39. $|\text{adj adj } A| =$

(a) $|A|^{2n}$ (b) $|A|^{n^2}$

(c) $|A|^{(n-1)^2}$ (d) $|A|^{(n+1)^2}$

40. adj adj A =

(a) A (b) $|A|^n A$

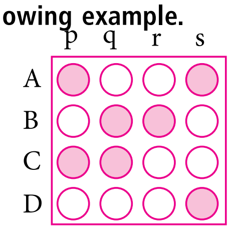
(c) $|A|^{n-1} \cdot A$ (d) $|A|^{n-2} \cdot A$

SECTION-III

(Matrix – Match Type)

This section contains questions. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example.

If the correct matches are A – p, s, B – q, r, C – p, q and D – s, then the correctly bubbled 4 × 4 matrix should be as follows :



41. Match the following :

	Column I		Column II
(A)	In a triangle ABC, if a, b, c are in $A.P.$, the maximum value of B is	(p)	$\frac{2\pi}{3}$
(B)	In a triangle ABC, if r_1, R, r_2 are in $A.P.$, then C is	(q)	$\frac{3\pi}{4}$

(C)	If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}$ = cos θ, then the maximum value of θ is	(r)	$\frac{\pi}{4}$
(D)	Let z and ω be complex numbers such that $\bar{z}+i\bar{\omega}=0$ and arg (z ω) = π, then arg z is	(s)	$\frac{\pi}{3}$
		(t)	$\frac{\pi}{2}$

42. Match the following:

	Column I		Column II
(A)	If $f(x) = \|x+1\| + \|x-1\| - \|x\| - 1$, then $\int_{-2}^{2} f(x)dx =$	(p)	1
(B)	If $\int_0^{x^2} t\, f(t)dt = x^5 - x^3$, then $f(1) =$	(q)	2
(C)	If $(x + 2y^3)\frac{dy}{dx} = y$, $y(1) = 1$, then $y(8) =$	(r)	3
(D)	If $I_1 = \int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sin x}$ and $I_2 = \int_0^1 \frac{\tan^{-1}x}{x}dx$ then $\frac{I_1}{I_2} =$	(s)	4

43. Match the following:

A and B are 2 events such that $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$. Then

	Column I		Column II
(A)	$P(A \cap B) \in$	(p)	$\left[\frac{4}{9}, 1\right]$
(B)	$P(A \cup B) \in$	(q)	$\left[\frac{2}{5}, \frac{9}{10}\right]$
(C)	$P(A/B) \in$	(r)	$\left[\frac{2}{3}, 1\right]$
(D)	$P(B/A) \in$	(s)	$\left[\frac{4}{15}, \frac{3}{5}\right]$

44. Match the following:

	Column I		Column II
(A)	The remainder when 17^{20} is divided by 81	(p)	60
(B)	The number of integer solutions of $x + y + z + u = 3$, $x \geq -2$, $y \geq -1$, $z \geq 0$, $u \geq 1$	(q)	56
(C)	The coefficient of x^2y in the expansion of $(1 + x + 2y)^5$	(r)	55
(D)	If $C_r = \binom{10}{r}$, then $\sum_{r=1}^{10} \frac{r \cdot C_r}{C_{r-1}}$	(s)	46

SOLUTIONS

PAPER-I

1. (a, c) : The range of $|\sin x| = [0, 1]$.

A = range of ln $|\sin x| = (-\infty, 0]$.

The range of ln $|x| = (-\infty, \infty)$.

B = range of sin (ln $|x|$) = $[-1, 1]$

$\therefore$ $A \cup B = (-\infty, 0] \cup [-1, 1] = (-\infty, 1]$;

$A \cap B = (-\infty, 0) \cap [-1, 1] = [-1, 0]$.

2. (b)

3. (a,b,c, d) : $f(2^-) = f(2) = f(2^+) = 35$

$\therefore$ $f(x)$ is continuous, $f'(x) = \begin{cases} 6x+12, -1 \leq x < 2 \\ -1, 2 < x \leq 3 \end{cases}$

$f'(2^-) = 24$, $f'(2^+) = -1$, $\therefore$ $f'(2)$ does not exist.

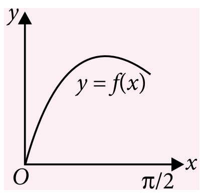
$f'(x) = 6(x + 2) > 0$ in $[-1, 2]$, $\therefore$ $f(x)$ increases in $[-1, 2]$

$\therefore$ $f(x)$ has maximum value 35 at $x = 2$.

4. (c) : Let $f(x) = \sin x + 2x - \frac{3x}{\pi}(x+1)$, $f(0) = 0$

$$f'(x) = \cos x + 2 - \frac{3}{\pi}(2x+1)$$

$$f'(0) = 3 - \frac{3}{\pi} > 0,\ f'\left(\frac{\pi}{2}\right) = -1 - \frac{3}{\pi} < 0$$



$$f''(x) = -\sin x - \frac{6}{\pi} < 0$$

$\therefore$ The function is convex and the graph of $y = f(x)$ lies above the x-axis.

Since $f\left(\frac{\pi}{2}\right) = \frac{\pi - 2}{4} > 0$

$\therefore$ $f(x) \geq 0 \Rightarrow \sin x + 2x \geq \frac{3x}{\pi}(x+1)$

5. (c) **6. (a, b, c)** **7. (d)**

8. (b, c): We have,

$2 - 2\cos\theta - 4\sin\theta + (1 - \cos 2\theta) + \sin 2\theta = 0$

$\Rightarrow\ 1 - \cos\theta - 2\sin\theta + \sin^2\theta + \sin\theta\cos\theta = 0$

$\Rightarrow\ (1 - \sin\theta)^2 - \cos\theta\,(1 - \sin\theta) = 0$

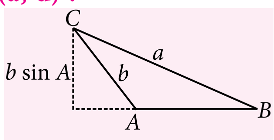
$\Rightarrow\ (1 - \sin\theta)\,(1 - \sin\theta - \cos\theta) = 0$

$$\Rightarrow\ 1 - \sin\theta = 0 \Rightarrow \theta = 2n\pi + \frac{\pi}{2} \quad \text{or} \quad \sin\theta + \cos\theta = 1$$

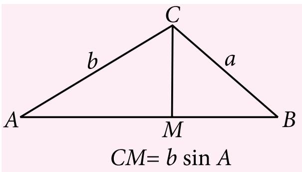
$$\Rightarrow\ \frac{\sin\theta}{\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\therefore\ \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4} \Rightarrow \theta = 2n\pi + \frac{\pi}{2}, 2n\pi$$

9. (a, d) :

(a) 

$$A > \frac{\pi}{2} \Rightarrow b\sin A < a$$

(d) 

One acute angled triangle exists.

10. (d): $z = r\,e^{i\theta},\ -\pi < \theta \le \pi$

$\Rightarrow z^i = r^i\,e^{-\theta} = e^{\ln r^i}.e^{-\theta}$

$= e^{(i\ln r - \theta)} = (\cos\ln r + i\sin\ln r)\,.\,e^{-\theta}$

$\Rightarrow |z^i| = |e^{-\theta}| \le e^{\pi}$,

Since $\theta > -\pi \Rightarrow -\theta \le \pi \Rightarrow e^{-\theta} \le e^{\pi}$.

11. (6)

12. (1): $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]^2 = 0$

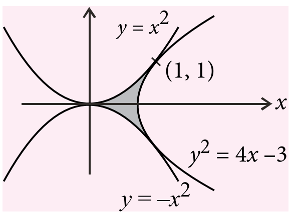
$$\Rightarrow \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} = 0$$

$\Rightarrow\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 1.$

13. (6): $\dfrac{ca}{rR} = \dfrac{ca}{r(abc)/4\Delta} = \dfrac{4\Delta}{rb} = \dfrac{4s}{b} = \dfrac{2\cdot 3b}{b} = 6$

14. (8) **15. (2)** **16. (4)**

17. (4): The curves touch at $(1, \pm 1)$ and $(0, 0)$



$$\text{The area } = 2\int_0^1 \left(\frac{y^2+3}{4} - \sqrt{y}\right) dy = \frac{5}{3} - \frac{4}{3} = \frac{1}{3} = \frac{m}{n}$$

$\Rightarrow\ m + n = 1 + 3 = 4.$

18. (9) **19. (3)** **20. (9)**

PAPER_II

21. (b): Since, $f(x) = \int f(x)dx$

$\therefore\ f'(x) = f(x)$ ...(i)

$$\therefore \int [f''(x) + \{f'(x)\}^2 + \{f(x)\}]^3 dx = 0$$

$$\Rightarrow\ f'(x) + \int f(x)\,f'(x)\,dx + \int \{f(x)\}^2 \cdot f(x)\,dx = 0$$

$$\Rightarrow\ f'(x) + \frac{\{f(x)\}^2}{2} + \int \{f(x)\}^2 f'(x)\,dx = 0$$

$$\Rightarrow\ f'(x) + \frac{\{f(x)\}^2}{2} + \frac{\{f(x)\}^3}{3} + C = 0$$

$$\therefore\ \frac{f(x)}{1} + \frac{\{f(x)\}^2}{2} + \frac{\{f(x)\}^3}{3} = -C, \text{ which is constant.}$$

22 (d) **23. (d)**

24. (c): $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$

$\Rightarrow\ (ax - b)^2 + (bx - c)^2 + (cx - d)^2 \le 0$

$\Rightarrow\ (ax - b)^2 + (bx - c)^2 + (cx - d)^2 = 0$

$$\Rightarrow\ \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = x \Rightarrow b^2 = ac$$

or $2\log b = \log a + \log c$ ....(i)

$$\text{Now, } \begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = \begin{vmatrix} 130 & 54 & \log a + \log c \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$

$$= 2\begin{vmatrix} 65 & 27 & \log b \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = 0 \qquad \text{(Using (i))}$$

25. (a) **26. (d)**

27. (a): We have

$P(B \cap C') = P[(A \cup A') \cap (B \cap C')]$

$$= P(A \cap B \cap C') + P(A' \cap B \cap C') = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\text{Now, } P(B \cap C) = P(B) - P(B \cap C') = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

28. (a) **29. (a)** **30. (a)**

31. (d): The d.r's, $<\alpha, \beta, \gamma>$ of L_1 are given by $\alpha + 2\beta - \gamma = 0$,

$$3\alpha-\beta+2\gamma=0\Rightarrow\frac{\alpha}{3}=\frac{\beta}{-5}=\frac{\gamma}{-7}$$

The d.r's of L_2 are given by $2\alpha - 2\beta + 3\gamma = 0$,

$\alpha - \beta + \gamma = 0 \Rightarrow \frac{\alpha}{1}=\frac{\beta}{1}=\frac{\gamma}{0}$

If θ is the angle between L_1 and L_2, then

$$\cos\theta=\frac{|3\times1-5\times1|}{\sqrt{3^2+5^2+7^2}\sqrt{1+1}}=\sqrt{\frac{2}{83}}$$

32. (b): L_1 is $x + 2y = 3 + z,\ 3x - y = 1 - 2z$ ...(i)

L_2 is $2x - 2y = 2 - 3z,\ x - y = -1 - z$ ...(ii)

(ii) $\Rightarrow 2 - 3z = -2 - 2z \Rightarrow z = 4$

$x + 2y = 7,\ 2x - 2y = -10 \Rightarrow x = -1, y = 4, z = 4$

The distance of the point $(-1, 4, 4)$ from the origin is $\sqrt{1+16+16}=\sqrt{33}$.

33. (c) : The plane through L_1 is

$x + 2y - z - 3 + \lambda(3x - y + 2z - 1) = 0$ ...(i)

The plane through L_2 is

$2x - 2y + 3z - 2 + \mu(x - y + z + 1) = 0$ .....(ii)

(i) and (ii) are same if $\frac{1+3\lambda}{2+\mu}=\frac{2-\lambda}{-|2+\mu|}$

$\Rightarrow\ 1+3\lambda=\lambda-2\ \Rightarrow\ \lambda=\frac{-3}{2}$

Now (i) gives $7x - 7y + 8z + 3 = 0$

The distance of $(0, 0, 0)$ from it is $\frac{3}{\sqrt{49+49+64}}$

$=\frac{3}{\sqrt{162}}=\frac{1}{3\sqrt{2}}.$

34. (c) : The normal at t, $xt + y = t^3 + 2t$ passes through $(3, 2)$

$\therefore\ t^3 - t - 2 = 0$. The roots t_1, t_2, t_3 satisfy

$\Sigma t_1 = 0,\ \Sigma t_1t_2 = -1,\ t_1t_2t_3 = 2$

$A = (t_2t_3,\ t_2 + t_3),\ B = (t_3t_1,\ t_3 + t_1),\ C = (t_1t_2,\ t_1 + t_2)$

The centroid $=G\left(\frac{\Sigma t_2t_3}{3},\frac{2\Sigma t_1}{3}\right)=\left(-\frac{1}{3},0\right)$

Its distance from the focus $(1, 0)$ is $\frac{4}{3}$.

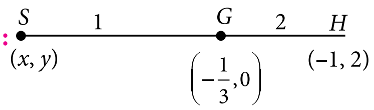
35. (d): The altitude from A is $\frac{y-(t_2+t_3)}{x-t_2t_3}=-t_1$

$\Rightarrow\ xt_1 + y = 2 + t_2 + t_3$

Likewise $xt_2 + y = 2 + t_3 + t_1 \Rightarrow x = -1, y = 2$

The distance of $H(-1, 2)$ from $(1, 0)$ is $2\sqrt{2}$.

36. (a) :



Since in a triangle S, G, H lies on a line and G divide SH in the ratio 1 : 2

$\therefore$ By section formula $S = (0, -1)$

Its distance from $(1, 0)$ is $\sqrt{2}$.

37. (c) : $A.\ \text{adj}\ A = |A|I$ ...(1)

Taking determinants, $|\text{adj}\ A| = \frac{|A|^n}{|A|}=|A|^{n-1}$.

38. (b): (1) $\Rightarrow (\text{adj}\ A)^{-1}=\frac{A}{|A|}$.

39. (c) : Since $|\text{adj}\ A| = |A|^{n-1}$ ...(2)

Replace A by adj A, we get

$|\text{adj}(\text{adi}\ A)| = |\text{adj}\ A|^{n-1}$

$= ((|A|)^{n-1})^{n-1}$ (Using (2))

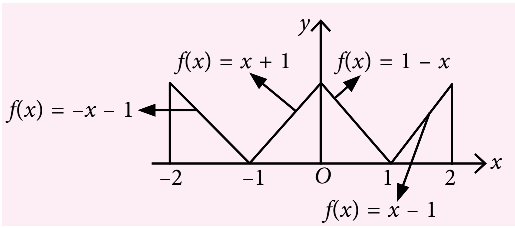
$= |A|^{(n-1)^2}$

40. (d) : (2) $\Rightarrow$ adj (adj A) $= |A|^{n-1}(\text{adj}\ A)^{-1} = |A|^{n-2}A$.

41. A $\to$ s ; B $\to$ t ; C $\to$ p ; D $\to$ q

42. A $\to$ q; B $\to$ p; C $\to$ q; D $\to$ q

$$f(x)=\begin{cases}-x-1 \text{ if } x<-1\\ x+1 \text{ if } -1<x<0\\ 1-x \text{ if } 0<x<1\\ x-1 \text{ if } x>1\end{cases}$$



(a) Area $=\int_{-2}^{2} f(x)dx=4\times\frac{1}{2}=2$

(b) Differentiating, $2x{\cdot}x^2f(x^2) = 5x^4 - 3x^2$

$x = 1 \Rightarrow f(1) = 1$

(c) $\frac{dx}{dy}-\frac{x}{y}=2y^2$

Linear d.e. with I.F. $=\frac{1}{y}$, solution is $\frac{x}{y}=\int 2y\,dy=y^2+c$

$y(1) = 1 \Rightarrow c = 0 \Rightarrow y = x^{1/3} \Rightarrow y(8) = 8^{1/3} = 2$

(d) $I_2=\int_0^1\frac{\tan^{-1}x}{x}dx$, put $x=\tan\theta$

$=\int_0^{\pi/4}\frac{2\theta\,d\theta}{\sin 2\theta}$, put $2\theta=x$

$\Rightarrow\ \frac{1}{2}\int_0^{\pi/2}\frac{x}{\sin x}dx=\frac{1}{2}I_1\ \therefore\ \frac{I_1}{I_2}=2$

43. A $\to$ s ; B $\to$ r; C $\to$ q; D $\to$ p

44. A $\to$ s ; B $\to$ q; C $\to$ p; D $\to$ r

JEE Advanced

PRACTICE PROBLEMS

Exam on 24th May

PAPER-1

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct.

1. If A and B are two matrices such that, $AB = BA$, then $\forall\ n \in N$,

(a) $A^nB = BA^n$ (b) $(AB)^n = A^nB^n$

(c) $(A+B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \ldots + {}^nC_n B^n$

(d) $A^{2n} - B^{2n} = (A^n - B^n)(A^n + B^n)$

2. $\displaystyle\int_0^{\pi/2} \left(\frac{1+\sin 3x}{1+2\sin x}\right) dx =$

(a) $\displaystyle\int_0^{\pi/2} \frac{1-\cos 3x}{1+2\cos x} dx$ (b) $\displaystyle\int_0^{\pi/2} \frac{\cos 3x+1}{2\cos x-1} dx$

(c) $\pi/2$ (d) 1

3. The function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ is

(a) continuous at $x = 1$

(b) differentiable at $x = 1$

(c) continuous at $x = 3$

(d) differentiable at $x = 3$

4. Let $f(x) = \sin(\pi x) - 4x(1-x)$, then

(a) $\dfrac{\sin \pi x}{x(1-x)} \le 4\ \forall\ x \in (0,1)$

(b) $f'\left(\dfrac{5}{8}\right) + f'\left(\dfrac{3}{8}\right) = 0$

(c) $f''(x) = 0$ has exactly two solutions in (0, 1)

(d) Rolle's theorem can be applied to $f'(x)$ in $[\alpha, 1/2]$ for some $\alpha \in (0, 1/2)$

5. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true?

(a) $P\left[X_1^c \mid X\right] = \dfrac{3}{16}$

(b) P[exactly two engines of the ship are functioning $\mid X$] = 7/8

(c) $P[X \mid X_2] = \dfrac{5}{16}$

(d) $P[X \mid X_1] = \dfrac{7}{16}$

6. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2 \dfrac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then

(a) $b + c = 4a$ (b) $b + c = 2a$

(c) locus of point A is a circle.

(d) locus of point A is an ellipse.

7. The equations of the line of shortest distance between the lines $\dfrac{x}{2} = \dfrac{y}{-3} = \dfrac{z}{1}$ and $\dfrac{x-2}{3} = \dfrac{y-1}{-5} = \dfrac{z+2}{2}$ are

(a) $3(x-21) = 3y + 92 = 3z - 32$

(b) $\dfrac{x-(62/3)}{1/3} = \dfrac{y+31}{1/3} = \dfrac{z-(31/3)}{1/3}$

(c) $\dfrac{x-21}{1/3} = \dfrac{y+(92/3)}{1/3} = \dfrac{z-(32/3)}{1/3}$

(d) $\dfrac{x-2}{1/3} = \dfrac{y+3}{1/3} = \dfrac{z-1}{1/3}$

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8. In the expression of $(x + y^2 + z^3)^9$
(a) the coefficient of $x^6y^4z^3$ is 252
(b) the coefficient of $x^4y^4z^6$ is 0
(c) the coefficient of $x^3y^8z^6$ is 1260
(d) all of these

9. Let x be the elements of the set $A = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$ and x_1, x_2, x_3 be positive integers and d be the number of integral solutions of $x_1x_2x_3 = x$ then d is divisible by
(a) 10 (b) 15 (c) 20 (d) 25

10. A twice differentiable function $f(x)$ is defined for all real numbers and satisfies the following conditions: $f(0) = 2, f'(0) = -5$ and $f''(0) = 3$. The function $g(x)$ is defined by $g(x) = e^{ax} + f(x)\ \forall\ x \in R$, where '$a$' is any constant. If $g'(0) + g''(0) = 0$ then a can be equal to
(a) 1 (b) –1 (c) 2 (d) –2

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive).

11. Let $\vec{a}$ and $\vec{b}$ be unit vectors. If $\vec{c}$ is a vector such that $\vec{c}+(\vec{c}\times\vec{a})=\vec{b}$, then the maximum value of $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ is $A \times 10^{-1}$. Find A.

12. If the circles which can be drawn, to pass through (1, 0) and (3, 0) and to touch the y-axis, intersect at an angle θ, then $4\cos\theta$ is equal to

13. If $\frac{3}{4!}+\frac{4}{5!}+\frac{5}{6!}+....+50 \text{ terms} = \frac{1}{3!}-\frac{1}{(K+3)!}$, then the last digit of sum of coefficients of $1 + 2^2x_1 + 3^2x_2 + ... + (23)^2x_{22}$ is

14. If $x > m, y > n, z > r, (x, y, z > 0)$ such that $\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix}=0$, then the greatest value of $\frac{27}{\left(1-\frac{m}{x}\right)\left(1-\frac{n}{y}\right)\left(1-\frac{r}{z}\right)}$ is

15. If both the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then find the value of b^2.

16. The points with co-ordinates $(a, b), (a_1, b_1), (a_2, b_2)$ are points of parabola $y = 3x^2$. The numbers a, a_1, a_2 are in A.P. and b, b_1, b_2 are in G.P. Calculate the ratio of the geometric sequence.

17. The value of $\int_{-2}^{2} |1-x^2|\,dx$ is

18. Through a point A on the axis a straight line is drawn parallel to y-axis so as to meet the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ in B and C. If $AB = BC$ and $8h^2 = kab$, then find the value of k.

19. The coefficient of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

20. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20$ =

PAPER-2

MULTIPLE CHOICE QUESTIONS

`This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. If $f(x + y) = f(x) \cdot f(y)$ for all x, y, z and $f(2) = 4$, $f'(0) = 3$, then $f'(2)$ equals
(a) 12 (b) 9 (c) 16 (d) 6

2. The coefficient of x^{10} in the expansion of $\sum_{r=0}^{20} (-1)^r\ {}^{20}C_r(x+1)^r\left(\frac{1}{2}+x\right)^{20-r}$, is
(a) 0 (b) 2^{10}
(c) ${}^{20}C_{10}$ (d) none of these

3. The general solution of the differential equation, $y' + y\phi'(x) - \phi(x)\cdot\phi'(x) = 0$, is
(a) $y = ce^{-\phi(x)} + \phi(x) - 1$
(b) $y = ce^{+\phi(x)} + \phi(x) - 1$
(c) $y = ce^{-\phi(x)} - \phi(x) + 1$
(d) $y = ce^{-\phi(x)} + \phi(x) + 1$
where c is an arbitrary constant.

4. If x and y are positive real number and m, n are any positive integers and $E=\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}$ then
(a) $E>\frac{1}{4}$ (b) $E>\frac{1}{2}$ (c) $E\le\frac{1}{4}$ (d) $E<\frac{1}{8}$

5. The function $f(x)=\log_e(1+x)-\frac{2x}{2+x}$ is increasing in

(a) $(-1, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)$ (d) none of these

6. $\int\left(\frac{\ln x-1}{(\ln x)^2+1}\right)^2 dx$ is equal to

(a) $\frac{x}{x^2+1}+c$ (b) $\frac{\ln x}{(\ln x)^2+1}$
(c) $\frac{x}{(\ln x)^2+1}+c$ (d) $e^x\left(\frac{x}{x^2+1}\right)+c$

7. If $[2\cos x] + [\sin x] = -3$, then the range of the function, $f(x)=\sin x+\sqrt{3}\cos x$ in $[0, 2\pi]$ is (where $[\cdot]$ denotes greatest integer function)

(a) $[-2, -1)$ (b) $(-2, -1]$
(c) $(-2, -1)$ (d) $[-2, -\sqrt{3})$

8. If the graph of the function $f(x)$ is symmetrical about two lines $x = a$ and $x = b$ then $f(x)$ must be periodic with period

(a) $\left|\frac{b-a}{2}\right|$ (b) $|b - a|$
(c) $|2(b - a)|$ (d) none of these

9. The coefficient of y^n in the polynomial $(y + {}^{2n+1}C_n)(y + {}^{2n+1}C_{n-1}) \dots (y + {}^{2n+1}C_0)$ is

(a) 2^{2n} (b) $2^{2n+1} - 1$
(c) $2n + 1$ (d) $2^{2n-1} + 1$

10. The slope of the normal at the point with abscissa $x = -2$ of the graph of the function $f(x) = |x^2 + x|$ is

(a) $-1/6$ (b) $-1/3$ (c) $1/6$ (d) $1/3$

PARAGRAPH TYPE

This section contains 3 paragraphs. Based upon each of the paragraphs 2 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Paragraph for Q. No. 11 and 12

It is given that $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$, where $x > 0$ and $B = (\cos^{-1}t)^2 + (\sin^{-1}t)^2$, where $t \in \left[0, \frac{1}{\sqrt{2}}\right]$.

11. The interval in which A lies is

(a) $\left[\frac{\pi^3}{7}, \frac{\pi^3}{2}\right)$ (b) $\left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right)$
(c) $\left[\frac{\pi^3}{40}, \frac{\pi^3}{10}\right)$ (d) none of these

12. The maximum value of B is

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{16}$
(c) $\frac{\pi^2}{4}$ (d) none of these

Paragraph for Q. No. 13 and 14

A straight line passing through $O(0, 0)$ cuts the lines $x = \alpha$, $y = \beta$ and $x + y = 8$ at A, B and C respectively such that $OA \cdot OB \cdot OC = 48\sqrt{2}$ and $f(\alpha, \beta) \le 0$, where $f(x, y)=\left|\frac{y}{x}-\frac{3}{2}\right|+(3x-2y)^6+\sqrt{e^x+2y-2e-6}$

13. The point of intersection of lines $x = \alpha$ and $y = \beta$ is

(a) $(3, 2)$ (b) $(2, 3)$ (c) $(3, 4)$ (d) $(e, 3)$

14. $OA + OB + OC$ equals

(a) $2\sqrt{2}$ (b) $4\sqrt{2}$ (c) $7\sqrt{2}$ (d) $9\sqrt{2}$

Paragraph for Q. No. 15 and 16

The range of a function $y = f(x)$ is the set of all possible output values $f(x)$ corresponding to every input x in the domain of f and is denoted as $f(A)$ if A is the domain. For finding the range of a function $y = f(x)$, first of all, find the domain of f.
If the domain consists of finite number of points, then the range consists of set of corresponding $f(x)$ values.
If the domain consists of whole real line or real line minus some finite points, then express x in term of y as $x = g(y)$. The values of y for which g is defined is the required range.
If the domain is a finite interval, find the intervals in which $f(x)$ increase/decrease and then find the extreme values of the function in those intervals. The union of those intervals is the required range.

15. Range of the function $f(x)=\frac{x^2+x+1}{x^2+1}$ is

(a) $(1, 3)$ (b) $\left[\frac{1}{2}, 3\right]$
(c) $\left[\frac{1}{2}, \frac{3}{2}\right]$ (d) $[0, 3]$

16. Range of $f(x) = [|\sin x| + |\cos x|]$, where $[\cdot]$ denotes the greatest integer function, is

(a) $\{1\}$ (b) $\{0, 1\}$
(c) $\{0\}$ (d) none of these

MATCHING LIST TYPE (ONLY ONE OPTION CORRECT)

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which one is correct.

17. Consider all possible permutations of letters of the word ENDEANOEL.

	Column I		Column II
(P)	The number of permutations containing the word ENDEA is	1.	5!
(Q)	The number of permutations in which the letter E occurs in the first and the last positions is	2.	2·5!
(R)	The number of permutations in which none of the letters D, L, N occurs in the last five positions is	3.	7·5!
(S)	The number of permutations in which the letters A, E, O occur only in odd positions is	4.	21·5!

	P	Q	R	S
(a)	1	4	2	2
(b)	3	4	2	2
(c)	1	3	2	2
(d)	3	1	2	3

18. Match the following:

	Column I		Column II
(P)	If 7^{103} is divided by 25, then the remainder is	1.	8
(Q)	The sum of rational terms in the expansion of $(\sqrt{2}+3^{1/5})^{10}$ is	2.	225
(R)	For all $n \in N$, $2^{4n} - 15n - 1$ is divisible by	3.	18
(S)	When 5^{99} is divided by 13, the remainder is	4.	41

	P	Q	R	S
(a)	4	1	2	3
(b)	3	4	2	1
(c)	1	3	2	4
(d)	3	1	2	4

19. Match the following:

	Column I		Column II
(P)	$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$	1.	lies in $3x + 2y + 6z - 12 = 0$
(Q)	$\frac{x+2}{2}=\frac{y+3}{3}=\frac{z-4}{-2}$	2.	is parallel to $2x + y - 2z = 3$
(R)	$\frac{x}{-2}=\frac{y}{-3}=\frac{z}{4}$	3.	is perpendicular to $4x + 7y + 6z = 0$
(S)	$\frac{x-1}{4}=\frac{y}{7}=\frac{z+1}{6}$	4.	passes through $(-2, -3, 4)$

	P	Q	R	S
(a)	1, 4	1, 3	2, 4	3
(b)	3	1	1, 2	4
(c)	1, 3	3, 4	2, 3	2
(d)	2	1, 4	4	3

20. Match the following :

	Column I		Column II
(P)	If $\lvert z_1\rvert = 12$ and $\lvert z_2 - 3 - 4i\rvert = 5$ then the minimum value of $\lvert z_1 - z_2\rvert$ is	1.	3
(Q)	If $\lim\limits_{x\to 0}\frac{(1-\cos x)(e^x-\cos x)}{x^n}$ is a non-zero finite number then the integer n is	2.	1
(R)	If $f(x)=\frac{1-x}{1+x}$ for $x > 0$ then the minimum value of $f\{f(x)\}+f\left\{f\left(\frac{1}{x}\right)\right\}$ is	3.	2
(S)	If z is a complex number satisfying $z\bar{z}-2(z+\bar{z})+3=0$ then the greatest value of $\lvert z\rvert$ is	4.	4

	P	Q	R	S
(a)	2	2	3	3
(b)	3	1	3	1
(c)	1	3	3	1
(d)	4	1	1	2

ANSWERS

PAPER-1

1. (a, b, c, d) 2. (a, b, d)
3. (a, b, c) 4. (a, b, c, d) 5. (b, d)
6. (b) 7. (a, b, c) 8. (d)
9. (a, c) 10. (a, d) 11. (5) 12. (2) 13. (6)
14. (8) 15. (7) 16. (3) 17. (4) 18. (9)
19. (6) 20. (5)

PAPER-2

1. (a) 2. (a) 3. (a) 4. (c) 5. (a)
6. (c) 7. (d) 8. (c) 9. (a) 10. (d)
11. (b) 12. (c) 13. (b) 14. (d) 15. (c)
16. (a) 17. (a) 18. (b) 19. (d) 20. (b)

For detailed solution to the Sample Paper, visit our website www.vidyalankar.org

CLASS XII

SOLVED PAPER CBSE BOARD 2015

Time Allowed : 3 hrs

Maximum Marks : 100

GENERAL INSTRUCTIONS

(i) All questions are compulsory.

(ii) Please check that this Question Paper contains 26 Questions.

(iii) Marks for each question are indicated against it.

(iv) Questions 1 to 6 in Section-A are Very Short Answer Type Questions carrying one mark each.

(v) Questions 7 to 19 in Section-B are Long Answer I Type Questions carrying 4 marks each.

(vi) Questions 20 to 26 in Section-C are Long Answer II Type Questions carrying 6 marks each.

(vii) Please write down the serial number of the Question before attempting it.

SECTION-A

1. If a line makes angles 90°, 60° and θ with x, y and z-axis respectively, where θ is acute, then find θ.

2. Write the element a_{23} of a 3×3 matrix $A = (a_{ij})$ whose elements a_{ij} are given by $a_{ij} = \frac{|i-j|}{2}$.

3. Find the differential equation representing the family of curves $v = \frac{A}{r} + B$, , where A and B are arbitrary constants.

4. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

5. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

6. Find λ, if the vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$ are coplanar.

SECTION-B

7. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black.

OR

A unbiased coin is tossed 4 times. Find the mean and variance of the number of heads obtained.

8. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find $(\vec{r} \times \hat{i}).(\vec{r} \times \hat{j}) + xy$

9. Find the distance between the point $(-1, -5, -10)$ and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

10. If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$, then find x.

OR

If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then find x.

11. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, $x^2 \le 1$ then find $\frac{dy}{dx}$.

12. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

13. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

14. Find : $\int (x+3)\sqrt{3-4x-x^2}\,dx$.

15. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below.

Article/School	A	B	C
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also, find the total funds collected for the purpose.
Write one value generated by the above situation.

16. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$

OR

If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

17. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, using properties of determinants find the value of $f(2x) - f(x)$.

18. Find : $\int \frac{dx}{\sin x + \sin 2x}$

OR

Integrate the following w.r.t. x.

$\frac{x^2 - 3x + 1}{\sqrt{1 - x^2}}$

19. Evaluate : $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

SECTION-C

20. Solve the differential equation :
$(\tan^{-1}y - x)dy = (1 + y^2)dx.$

OR

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ given that $y = 1$, when $x = 0$.

21. If lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines.

22. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then find $P(A)$ and $P(B)$.

23. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values.

24. Find graphically, the maximum value of $z = 2x + 5y$, subject to constraints given below:

$2x + 4y \le 8$
$3x + y \le 6$
$x + y \le 4$
$x \ge 0, y \ge 0$

25. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b)\ R\ (c, d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.

26. Using integration find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

OR

Evaluate : $\int_1^3 (e^{2-3x} + x^2 + 1)dx$ as a limit of a sum.

SOLUTIONS

1. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2 90° + \cos^2 60° + \cos^2 \theta = 1$

$\Rightarrow \frac{1}{4} + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{3}{4}$

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$ $(\because \theta$ is acute)

$\Rightarrow \theta = \frac{\pi}{6}$

2. $a_{23} = \frac{|2-3|}{2} = \frac{1}{2}$

3. Given, $v = \frac{A}{r} + B$

$\Rightarrow \frac{dv}{dr} = -\frac{A}{r^2} \Rightarrow \frac{d^2v}{dr^2} = \frac{2A}{r^3}$

$\Rightarrow \frac{d^2v}{dr^2} \div \frac{dv}{dr} = \frac{2A}{r^3} \div \left(\frac{-A}{r^2}\right)$

or, $\frac{d^2v}{dr^2} \div \frac{dv}{dr} = \frac{-2}{r}$

or, $\frac{d^2v}{dr^2} = \frac{-2}{r} \cdot \frac{dv}{dr}$

$\Rightarrow \dfrac{d^2v}{dr^2}+\dfrac{2}{r}\dfrac{dv}{dr}=0$ is the required differential equation.

4. We have, $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy}=1$

or $\dfrac{dy}{dx}+\dfrac{1}{\sqrt{x}}y=\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$ ...(i)

Compare (i) with linear differential equation

$\dfrac{dy}{dx}+Py=Q$

I.F. $= e^{\int P dx}$

Hence, I.F $= e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

5. Projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$=\dfrac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}}$

$=\dfrac{14+6-12}{7}=\dfrac{8}{7}$

6. Since the vectors are coplanar

$\therefore \begin{vmatrix}1 & 3 & 1\\ 2 & -1 & -1\\ 0 & \lambda & 3\end{vmatrix}=0$

$\Rightarrow$ $1(-3+\lambda)-3(6-0)+1(2\lambda-0)=0$

$\Rightarrow$ $-3+\lambda-18+2\lambda=0$

$\Rightarrow$ $3\lambda-21=0$

$\Rightarrow$ $\lambda=7$

7. Probability of choosing the bag A

$= P(A)=\dfrac{2}{6}=\dfrac{1}{3}$

Probability of choosing the bag $B=P(B)=\dfrac{4}{6}=\dfrac{2}{3}$

Let E_1 and E_2 be the events of drawing a red and black ball from bag A and B resp.

$\therefore$ $P(E_1)=\dfrac{6\times 4}{^{10}C_2}$ and $P(E_2)=\dfrac{7\times 3}{^{10}C_2}$

$\therefore$ Required probability

$= P(A)\times P(E_1)+P(B)\times P(E_2)$

$=\dfrac{1}{3}\times\dfrac{6\times 4}{^{10}C_2}+\dfrac{2}{3}\times\dfrac{7\times 3}{^{10}C_2}$

$=\dfrac{8}{45}+\dfrac{14}{45}=\dfrac{22}{45}$

OR

Let p and q be the respective probabilities of occuring a head and tail in single thrown of a coin.

Then $p=q=\dfrac{1}{2}$

x_i	0	1	2	3	4
$p(x_i)$	$^4C_0\,p^0q^4$	$^4C_1\,pq^3$	$^4C_2\,p^2q^2$	$^4C_3\,p^3\,q^1$	$^4C_4\,p^4$

Mean $=\sum x_i\,p(x_i)$

$$=\left[0\times {^4C_0}\times\left(\frac{1}{2}\right)^4+{^4C_1}\left(\frac{1}{2}\right)^4+2\times{^4C_2}\left(\frac{1}{2}\right)^4+3\times{^4C_3}\left(\frac{1}{2}\right)^4+4\times{^4C_4}\left(\frac{1}{2}\right)^4\right]$$

$$=\left(\frac{1}{2}\right)^4\times[4+12+12+4]=\frac{32}{16}=2$$

Variance $=\sum x_i^2\,p(x_i)-\left(\sum x_i p(x_i)\right)^2$

$$=\left[0\times\left(\frac{1}{2}\right)^4+{^4C_1}\left(\frac{1}{2}\right)^4+4\times{^4C_2}\left(\frac{1}{2}\right)^4+9\times{^4C_3}\left(\frac{1}{2}\right)^4+16\times{^4C_4}\left(\frac{1}{2}\right)^4\right]-(2)^2$$

$$=\left(\frac{1}{2}\right)^4[4+24+36+16]-4=1$$

8. $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy$

$=((x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i})\cdot((x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j})+xy$

$=(-y\hat{k}+z\hat{j})\cdot(x\hat{k}-z\hat{i})+xy$

$=-xy+xy=0$

9. Equation of given line is

$\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z-2}{12}=k\text{(say)}$

$\Rightarrow$ $x=3k+2,\ y=4k-1,\ z=12k+2$

Since pt. $(3k+2, 4k-1, 12k+2)$ lie on plane

$x-y+z=5$

$\Rightarrow$ $3k+2-4k+1+12k+2=5$

$\Rightarrow$ $11k=0$ $\Rightarrow$ $k=0$

point is $(2, -1, 2)$

Required distance $=\sqrt{(2+1)^2+(5-1)^2+(2+10)^2}$

$=\sqrt{9+16+144}=\sqrt{169}=13$

10. $\sin[\cot^{-1}(x+1)]=\cos(\tan^{-1}x)$

Let $\cot^{-1}(x+1)=A$

$\Rightarrow$ $x+1=\cot A$

$\Rightarrow$ $\sin A=\dfrac{1}{\sqrt{(x+1)^2+1}}$ and let $\tan^{-1}x=B$

$\Rightarrow \quad x = \tan B$

$$\Rightarrow \quad \cos B = \frac{1}{\sqrt{x^2+1}}$$

$\Rightarrow \quad \sin A = \cos B$

$$\Rightarrow \quad \frac{1}{\sqrt{(x+1)^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

$\Rightarrow \quad (x + 1)^2 + 1 = x^2 + 1$

$\Rightarrow \quad 1 + 2x = 0$

$$\Rightarrow \quad x = -\frac{1}{2}$$

OR

$$(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad (\tan^{-1} x)^2 + \left(\frac{\pi}{2} - \tan^{-1} x\right)^2 = \frac{5\pi^2}{8}$$

Putting $\tan^{-1}x = \theta$

$$\Rightarrow \quad \theta^2 + \left(\frac{\pi}{2} - \theta\right)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \theta^2 + \frac{\pi^2}{4} + \theta^2 - \pi\theta = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad 2\theta^2 - \pi\theta + \left(\frac{\pi^2}{4} - \frac{5\pi^2}{8}\right) = 0$$

$$\Rightarrow \quad 2\theta^2 - \pi\theta - \frac{3}{8}\pi^2 = 0$$

$\Rightarrow \quad 16\theta^2 - 8\pi\theta - 3\pi^2 = 0$

$\Rightarrow \quad 4\theta(4\theta - 3\pi) + \pi(4\theta - 3\pi) = 0$

$\Rightarrow \quad (4\theta + \pi)\,(4\theta - 3\pi) = 0$

$\Rightarrow \quad$ either $4\theta = 3\pi$ or $4\theta = -\pi$

$$\Rightarrow \quad \theta = \frac{3\pi}{4} \quad \text{or} \quad \Rightarrow \quad \theta = -\frac{\pi}{4}$$

Hence, $\tan^{-1} x = \dfrac{3\pi}{4}$ or $-\dfrac{\pi}{4}$

$\Rightarrow \quad x = -1$

11. $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right), x^2 \le 1$

Put $x^2 = \cos\theta$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right) = \tan^{-1}\left(\frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right) = \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \quad y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)$$

Differentiate w.r.t 'x' on both sides, we get

$$\frac{dy}{dx} = -\frac{1 \times 2x}{2\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

12. Given, $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$

$\Rightarrow \quad x^2 = a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta$ ...(i)

and $y^2 = a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta$ ...(ii)

Adding (i) and (ii) gives

$x^2 + y^2 = a^2 + b^2$

Differentiate w.r.t 'x', we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0 \quad \Rightarrow \quad y\frac{dy}{dx} = -x \quad \text{...(iii)}$$

Again differentiate w.r.t. 'x'

$$\Rightarrow \quad 1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Multiply by 'y' on both sides

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2} + \left(y.\frac{dy}{dx}\right)\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

13. Let 'a' be side of an equilateral triangle

then $\dfrac{da}{dt} = 2$

Let 'A' be area of equilateral triangle, then $A = \dfrac{\sqrt{3}}{4}a^2$

$$\Rightarrow \quad \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4}a\frac{da}{dt}$$

$$= \frac{\sqrt{3}}{2} \times 20 \times 2$$

$$= 20\sqrt{3}\,\text{cm}^2/\text{sec}$$

14. Let $I = \int (x+3)\sqrt{3-4x-x^2}\,dx$

$$= \int (x+2+1)\sqrt{3-4x-x^2}\,dx$$

$$= \frac{-1}{2}\int -2(x+2)\sqrt{3-4x-x^2}\,dx + \int \sqrt{3-4x-x^2}\,dx$$

Let $I = I_1 + I_2$

$$I_1 = -\frac{1}{2}\int -2(x+2)\sqrt{3-4x-x^2}\,dx$$

Put $3 - 4x - x^2 = t \Rightarrow (-4 - 2x)dx = dt$

$$I_1 = -\frac{1}{2}\int \sqrt{t}\,dt = -\frac{1}{2} \times \frac{2}{3}(t)^{3/2} + c_1$$

$$= -\frac{1}{3} \times (3-4x-x^2)^{3/2} + c_1 \quad \text{...(i)}$$

$$I_2 = \int \sqrt{3-4x-x^2}\,dx$$
$$= \int \sqrt{-(x^2+4x-3+4-4)}\,dx$$
$$= \int \sqrt{-((x+2)^2-7)}\,dx$$
$$= \int \sqrt{7-(x+2)^2}\,dx$$
$$= \frac{(x+2)\sqrt{3-4x-x^2}}{2} + \frac{7}{2}\sin^{-1}\frac{(x+2)}{\sqrt{7}} + c_2 \quad ..(ii)$$

From (i) & (ii), we get

$$I = -\frac{1}{3}(3-4x-x^2)^{3/2} + \frac{(x+2)\sqrt{3-4x-x^2}}{2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

15. The number of articles sold by each school can be written in the matrix form as

$$X = \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

The cost of each article can be written in the matrix form as

$Y = [25 \quad 100 \quad 50]$

The fund collected by each school is given by

$$YX = [25 \quad 100 \quad 50]\begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

$= [7000 \quad 6125 \quad 7875]$

Thus, the funds collected by schools A, B and C are ₹ 7000

The total fund collected = ₹ (7000 + 6125 + 7875)

= ₹ 21000

The situation highlights the helped nature of the students.

16. $$A^2 - 5A + 4I = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & - & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 2 \\ 9 & 2 & 5 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Hence $X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$

OR

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$|A'| = 1(-1-8) - 2(-8+3)$

$= -9 + 10 = 1$

Let the cofactors of a_{ij}'s are C_{ij}

$C_{11} = (-1)^2(-1-8) = -9$

$C_{12} = (-1)^3(-2-6) = -8$

$C_{13} = (-1)^4(-8+3) = -5$

$C_{21} = (-1)^3(0+8) = -8$

$C_{22} = (-1)^4(1+6) = 7$

$C_{23} = (-1)^5(4-0) = -4$

$C_{31} = (-1)^4(0-2) = -2$

$C_{32} = (-1)^5(2-4) = 2$

$C_{23} = (-1)^6(-1-0) = -1$

$$adj(A') = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore \quad (A')^{-1} = \frac{adj(A')}{|A'|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

17. $$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$$

$$\Rightarrow \quad f(x) = a\begin{vmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, we get

$$f(x)=a\begin{vmatrix}1&0&0\\x&x+a&-1\\x^2&x^2+ax&a\end{vmatrix}$$

$\Rightarrow\ f(x) = a[a(x + a) + (x^2 + ax)]$

$\Rightarrow\ f(x) = a(a^2 + ax + ax + x^2)$

$= a(a^2 + 2ax + x^2)$

Also, $$f(2x)=\begin{vmatrix}a&-1&0\\2ax&a&-1\\4ax^2&2ax&a\end{vmatrix}$$

$$\Rightarrow\ f(2x)=a\begin{vmatrix}1&-1&0\\2x&a&-1\\4x^2&2ax&a\end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, we get

$$f(2x)=a\begin{vmatrix}1&0&0\\2x&2x+a&-1\\4x^2&4x^2+2ax&a\end{vmatrix}$$

$\Rightarrow\ f(2x) = a\{a(2x + a) + 4x^2 + 2ax\}$

$= a(4x^2 + a^2 + 4ax)$

$\therefore\ f(2x) - f(x) = a(4x^2 + a^2 + 4ax - a^2 - 2ax - x^2)$

$= ax(3x + 2a)$

18. $I=\int\frac{1}{\sin x+\sin 2x}dx$

$=\int\frac{1}{\sin x+2\sin x\cos x}dx$

$=\int\frac{1}{\sin x(1+2\cos x)}dx$

$=\int\frac{\sin x}{\sin^2 x(1+2\cos x)}dx$

Let $u = \cos x \Rightarrow du = -\sin x dx$

Also, $\sin^2x = 1-\cos^2x = 1-u^2$

$\therefore\ I=\int\frac{-1}{(1-u^2)(1+2u)}du$

$=\int\frac{-1}{(1+u)(1-u)(1+2u)}du$

Using partial fractions, we get

$$\frac{-1}{(1+u)(1-u)(1+2u)}=\frac{A}{(1+u)}+\frac{B}{(1-u)}+\frac{C}{(1+2u)}$$

$\Rightarrow\ -1 = A(1 - u)(1 + 2u) + B(1 + u)(1 + 2u) + C(1 + u)(1 - u)$

Put $u = 1$

$-1 = B(1 + 1)(1 + 2)$

$\Rightarrow\ -1 = 6B\ \Rightarrow\ B = -1/6$

put $u = -1$

$-1 = A(1 + 1)(1 - 2)$

$\Rightarrow\ -1 = -2A\ \Rightarrow\ A = 1/2$

put $u=-\frac{1}{2}$

$-1=C\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)$

$\Rightarrow\ -1=C\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)$

$\Rightarrow\ C=\frac{-4}{3}$

So, $$\frac{-1}{(1+u)(1-u)(1+2u)}=\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}$$

$\Rightarrow\ I=\int\left[\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}\right]du$

$=\frac{1}{2}\log(1+u)+\frac{1}{6}\log(1-u)-\frac{4}{3\times 2}\log(1+2u)+C$

$=\frac{1}{2}\log(1+\cos x)+\frac{1}{6}\log(1-\cos x)-\frac{2}{3}\log(1+2\cos x)+C$

OR

Let $I=\int\frac{x^2-3x+1}{\sqrt{1-x^2}}dx$

$=-\int\frac{-x^2+3x-1}{\sqrt{1-x^2}}dx$

$=-\int\frac{(1-x^2)+3x-2}{\sqrt{1-x^2}}dx$

or, $I=-\int\sqrt{1-x^2}\,dx+\int\frac{-3x+2}{\sqrt{1-x^2}}dx$

$=-\int\sqrt{1-x^2}\,dx+\frac{3}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1-x^2}}dx$

$=-\int\sqrt{1-x^2}\,dx+\frac{3\times 2}{2}\sqrt{1-x^2}+2\sin^{-1}x+C$

$=-\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]+3\sqrt{1-x^2}+2\sin^{-1}x+C$

$=-\frac{x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+3\sqrt{1-x^2}+C$

19. Let $I=\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2\,dx$

$$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx)\, dx$$

$$= \int_{-\pi}^{\pi} \cos^2 ax\, dx + \int_{-\pi}^{\pi} \sin^2 bx\, dx - 2\int_{-\pi}^{\pi} \cos ax \sin bx\, dx$$

Using property,

$$\int_{-a}^{a} f(x)dx = \begin{bmatrix} 0, \text{ if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx, \textit{ if } f(x) \text{ is even} \end{bmatrix}$$

$$= 2\left[\int_0^{\pi} \cos^2 ax\, dx + \int_0^{\pi} \sin^2 bx\, dx\right]$$

$$= 2\left[\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx + \int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx\right]$$

$$= \int_0^{\pi}(1+\cos 2ax)dx + \int_0^{\pi}(1-\cos 2bx)dx$$

$$= \left|2x\right|_0^{\pi} + \frac{1}{2a}\left|\sin 2ax\right|_0^{\pi} - \frac{1}{2b}\left|\sin 2bx\right|_0^{\pi}$$

$$= 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$$

20. We have, $(\tan^{-1}y - x)dy = (1 + y^2)dx$

or, $\dfrac{dx}{dy} = \dfrac{\tan^{-1} y - x}{1+y^2}$

or, $\dfrac{dx}{dy} + \dfrac{1}{1+y^2} \cdot x = \dfrac{\tan^{-1} y}{1+y^2}$

This is linear differential equation of form

$$\frac{dx}{dy} + Px = Q$$

where I.F. $= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$

Required solution is

$$x \cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + C$$

Put $\tan^{-1}y = t \Rightarrow \dfrac{1}{1+y^2}dy = dt$

or, $x \cdot e^{\tan^{-1} y} = \int t\, e^t dt + C$

or, $x \cdot e^{\tan^{-1} y} = t\, e^t - e^t + C$

or, $x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$

or, $x = \tan^{-1} y - 1 + \dfrac{C}{e^{\tan^{-1} y}}$

OR

Given differential equation is

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\Rightarrow\ v + x\dfrac{dv}{dx} = \dfrac{v}{1+v^2}$

or, $x\dfrac{dv}{dx} = \dfrac{v}{1+v^2} - v$

or, $x\dfrac{dv}{dx} = \dfrac{-v^3}{1+v^2}$

or, $\dfrac{dx}{x} = -\left(\dfrac{1+v^2}{v^3}\right)dv$

Integrating both sides

or, $\int \dfrac{dx}{x} = -\int v^{-3}dv - \int \dfrac{1}{v}dv$

or, $\log x = \dfrac{1}{2v^2} - \log v + C$

or, $\log x = \dfrac{x^2}{2y^2} - \log y + \log x + C$

or, $\log y = \dfrac{x^2}{2y^2} + C$ .....(i)

when $y = 1, x = 0$

$\Rightarrow \log 1 = 0 + C \quad \Rightarrow \quad C = 0$

Put in (i), we get

Particular solution is $y = e^{\frac{x^2}{2y^2}}$

21. We have, $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4} = \lambda$ (say)

or, $x = 2\lambda + 1, y = 3\lambda - 1, z = 4\lambda + 1$

Since, given both lines intersect

$\therefore$ point $(2\lambda + 1, 3\lambda - 1, 4\lambda + 1)$ satisfies

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$$

$\Rightarrow \dfrac{2\lambda+1-3}{1} = \dfrac{3\lambda-1-k}{2} = \dfrac{4\lambda+1}{1}$ .....(i)

$\Rightarrow 2\lambda - 2 = 4\lambda + 1 \Rightarrow 2\lambda = -3 \Rightarrow \lambda = \dfrac{-3}{2}$

Put value of λ in (i)

$$2\lambda - 2 = \frac{3\lambda - 1 - k}{2}$$

$$\Rightarrow \quad 2\left(\frac{-3}{2}\right)-2=\frac{3\left(\frac{-3}{2}\right)-1-k}{2}$$

$$\Rightarrow \quad -5\times 2=-\frac{11}{2}-k$$

$$\Rightarrow \quad k=\frac{-11}{2}+10=\frac{9}{2}$$

$\therefore$ Required equation of plane is

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

or, $(x-1)(-5)-(y+1)(2-4)+(z-1)(4-3)=0$

or, $-5x+5+2y+2+z-1=0$

or, $5x-2y-z-6=0$

22. It is given that A and B are independent events.

and $P(\bar{A}\cap B)=\frac{2}{15}$

$$\Rightarrow \quad P(\bar{A})P(B)=\frac{2}{15} \quad \text{...(i)}$$

Also, $P(A\cap \bar{B})=\frac{1}{6}$

$$\Rightarrow \quad P(A)P(\bar{B})=\frac{1}{6}$$

$$\Rightarrow \quad P(A)=\frac{1}{6[1-P(B)]} \quad \text{...(ii)}$$

From (i), we have

$$[1-P(A)]P(B)=\frac{2}{15}$$

or, $\left[1-\frac{1}{6[1-P(B)]}\right]P(B)=\frac{2}{15}$ (from (ii))

or, $\left\{\frac{6-6P(B)-1}{6[1-P(B)]}\right\}P(B)=\frac{2}{15}$

or, $5P(B)-6[P(B)]^2=\frac{12[1-P(B)]}{15}$

or, $25P(B)-30[P(B)]^2=4-4P(B)$

or, $30[P(B)]^2-29P(B)+4=0$

or, $30[P(B)]^2-24P(B)-5P(B)+4=0$

or, $6P(B)[5P(B)-4]-1[5P(B)-4]=0$

or, $[5P(B)-4][6P(B)-1]=0$

$$\Rightarrow \quad P(B)=\frac{4}{5},\frac{1}{6}$$

For $P(B)=\frac{4}{5}$, using (ii), we have

$$P(A)=\frac{1}{6[1-P(B)]}=\frac{1}{6\left[1-\frac{4}{5}\right]}=\frac{5}{6}$$

For $P(B)=\frac{1}{6}$, using (ii), we have

$$P(A)=\frac{1}{6\left[1-\frac{1}{6}\right]}=\frac{1}{5}$$

$\therefore \quad P(A)=\frac{5}{6}, P(B)=\frac{4}{5}$ or $P(A)=\frac{1}{5}, P(B)=\frac{1}{6}$

23. $f(x)=\sin x-\cos x$

$\Rightarrow \quad f'(x)=\cos x+\sin x$

Put $f'(x)=0$

$\Rightarrow \quad \cos x+\sin x=0$

$\Rightarrow \quad \tan x=-1 \quad \Rightarrow \quad x=\frac{3\pi}{4},\frac{7\pi}{4}$

$f''(x)=-\sin x+\cos x$

At $x=\frac{3\pi}{4}$, $f''(x)=-\sin\frac{3\pi}{4}+\cos\frac{3\pi}{4}$

$$=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=\frac{-2}{\sqrt{2}}=-\sqrt{2}$$

At $x=\frac{7\pi}{4}$, $f''(x)=-\sin\frac{7\pi}{4}+\cos\frac{7\pi}{4}$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Since $(f''(x))_{x=\frac{3\pi}{4}}<0$

$\therefore \quad f(x)$ has local maxima at $x=\frac{3\pi}{4}$

Since $(f''(x))_{x=\frac{7\pi}{4}}>0$

$\therefore \quad f(x)$ has local minima at $x=\frac{7\pi}{4}$

$\therefore \quad$ Local maximum value at $x=\frac{3\pi}{4}$ is

$$(f(x))_{x=\frac{3\pi}{4}}=\sin\frac{3\pi}{4}-\cos\frac{3\pi}{4}$$

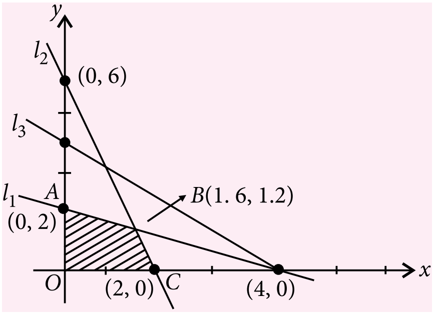
$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Local minimum value at $x=\frac{7\pi}{4}$ is

$$(f(x))_{x=\frac{7\pi}{4}}=\sin\frac{7\pi}{4}-\cos\frac{7\pi}{4}$$

$$=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=\frac{-2}{\sqrt{2}}=-\sqrt{2}$$

24. Let $l_1 : 2x + 4y = 9$
$l_2 : 3x + y = 6$
$l_3 : x + y = 4$
$l_4 : x = 0,\ l_5 : y = 0$



Shaded portion $OABC$ is the feasible region, where co-ordinates of the corner points are $O(0, 0)$, $A(0, 2)$, $B(1.6, 1.2)$, $C(2, 0)$
For B : solving l_2 & l_1, we get $B(1.6, 1.2)$
The value of objective function at these points are.

Points	Value of the objective function $z = 2x + 5y$
$O(0, 0)$	$2 \times 0 + 5 \times 0 = 0$
$A(0, 2)$	$2 \times 0 + 5 \times 2 = 10$
$B(1.6, 1.2)$	$2 \times 1.6 + 5 \times 1.2 = 9.2$
$C(2, 0)$	$2 \times 2 + 5 \times 0 = 4$

Out of these values of z, the maximum value of z is 10, which is at $A(0, 2)$.
Hence, the maximum value of z is 10 at $A(0, 2)$.

25. Reflexivity
Let (a, b) be an arbitrary element of $N \times N$. Then,
$(a, b) \in N \times N$
$\Rightarrow\ a, b \in N$
$\Rightarrow\ ab(b + a) = ba(a + b)$
[by comm. of add. and mult. on N]
$\Rightarrow\ (a, b)\ R\ (a, b)$
Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$.
So, R is reflexive on $N \times N$.

Symmetry
Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$.
Then, $(a, b)\ R\ (c, d)$
$\Rightarrow\ ad(b + c) = bc(a + d)$
$\Rightarrow\ cb(d + a) = da(c + b)$
[by comm. of add. and mult. on N]
Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$.
So, R is symmetric on $N \times N$.

Transitivity
Let $(a, b), (c, d), (e, f) \in N \times N$ be such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,
$(a, b)\ R\ (c, d) \Rightarrow\ ad(b + c) = bc(a + d)$

$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad} \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \quad \text{...(i)}$$

and, $(c, d)\ R\ (e, f) \Rightarrow\ cf(d + e) = de(c + f)$

$$\Rightarrow \frac{d+e}{de} = \frac{c+f}{cf} \Rightarrow \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$$\Rightarrow \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \Rightarrow \frac{b+e}{be} = \frac{a+f}{af}$$

$\Rightarrow\ af(b + e) = be(a + f)\ \Rightarrow (a, b)\ R\ (e, f)$
Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$
$\Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.
So, R is transitive on $N \times N$.
Hence, R is an equivalence relation.

26. Given equation of circle is $x^2 + y^2 = 4$
Differentiate w.r.t. 'x' on both sides

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1, \sqrt{3})} = \frac{-1}{\sqrt{3}}$$

Equation of tangent at $(1, \sqrt{3})$ is

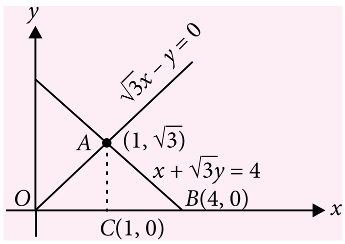
$$y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x - 1)$$

$$\Rightarrow \sqrt{3}y - 3 = -x + 1 \Rightarrow x + \sqrt{3}y = 4$$

Equation of normal at $(1, \sqrt{3})$ is

$$y - \sqrt{3} = \sqrt{3}(x - 1) \Rightarrow \sqrt{3}x - y = 0$$

Since point of intersection of $x + \sqrt{3}y = 4$ and $\sqrt{3}x - y = 0$ is $(1, \sqrt{3})$



Required area is area of ΔOAB

$$\Rightarrow \text{Area of } \Delta OAB = \int_0^1 \sqrt{3}x\,dx + \int_1^4 \left(\frac{4-x}{\sqrt{3}}\right)dx$$

Contd. on page no. 70

JEE ADVANCED : PRACTICE PAPER

READY STEADY GO!!

EXAM ON 24th MAY

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SECTION-I

(Single Correct Answer Type)

This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. The value of b such that the equation $\dfrac{b\cos x}{2\cos 2x-1}=\dfrac{b+\sin x}{(\cos^2 x-3\sin^2 x)\tan x}$ possesses solutions, belong to the set

(a) $\left(-\infty,\frac{1}{2}\right)$ (b) $\left(\frac{1}{2},\infty\right)$

(c) $(-\infty,\infty)$ (d) $\left(-\infty,-\frac{1}{2}\right)\cup(1,\infty)$

2. If m and n $(> m)$ are positive integers, the number of solutions of the equation $n|\sin x| = m|\cos x|$ in $[0, 2\pi]$ is

(a) m (b) n (c) mn (d) 4

3. If $\alpha=\dfrac{2\pi}{7}$, then the value of

$\tan\alpha \tan 2\alpha + \tan 2\alpha \tan 4\alpha + \tan 4\alpha \tan \alpha$ is

(a) -7 (b) -4 (c) 0 (d) 4

4. If $\sin x+\cos y=a$ and $\cos x+\sin y=b$, then $\tan\dfrac{x-y}{2}$ is equal to

(a) $a + b$ (b) $a - b$

(c) $\dfrac{a+b}{a-b}$ (d) $\dfrac{a-b}{a+b}$

5. If $\log\left(\dfrac{a+b}{3}\right)=\dfrac{\log a+\log b}{2}$, then $\dfrac{a}{b}+\dfrac{b}{a}$ is equal to

(a) 1 (b) 3 (c) 5 (d) 7

6. $\log_{10}\tan1° + \log_{10}\tan2° + ... + \log_{10}\tan89°$ is equal to

(a) 0 (b) 1 (c) 27 (d) 81

7. The value of

$$\left\{\frac{(81)^{\frac{1}{\log_5 9}}+3^{\left(\frac{3}{\log_{\sqrt{6}} 3}\right)}}{409}\right\}\left\{(\sqrt{7})^{\frac{2}{\log_{25} 7}}-(125)^{\log_{25} 6}\right\}$$ is

(a) 0 (b) 1 (c) 2 (d) 3

8. If $\log_{12}6 = x$, $\log_{24}54 = y$, then the value of $xy + (5x - 2y) + 4$ is

(a) 6 (b) 7 (c) 8 (d) 9

9. The sum of the series

$$\frac{1}{\sin 45°\sin 46°}+\frac{1}{\sin 47°\sin 48°}+....+\frac{1}{\sin 133°\sin 134°}$$

is cosec $n°$, then the integer n is

(a) 2 (b) 1 (c) 3 (d) 4

10. If $\cos A=\dfrac{3}{4}$, then the value of $32\sin\dfrac{A}{2}\sin\dfrac{5A}{2}$ is

(a) 11 (b) 12 (c) 13 (d) 14

11. If $x\in\left(0,\dfrac{\pi}{2}\right)$, one solution of $\dfrac{\sqrt{3}-1}{\sin x}+\dfrac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$ is $\dfrac{\pi}{12}$, the other solution must be $\dfrac{\lambda\pi}{36}$, then λ is

(a) 11 (b) 12 (c) 13 (d) 14

12. The smallest positive x $(y > 0)$ satisfying $x-y=\dfrac{\pi}{4}$, $\cot x + \cot y = 2$ is $\dfrac{5\pi}{\lambda}$, the numerical integer λ is

(a) 15 (b) 14 (c) 12 (d) 13

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He trains IIT and Olympiad aspirants.

13. The solution set of x in $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1 \geq 0$ is

(a) $\left[0, \frac{\pi}{6}\right] \cup \left\{\frac{\pi}{2}\right\} \cup \left[\frac{5\pi}{6}, \pi\right]$

(b) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ (c) $\left(\frac{5\pi}{6}, \pi\right)$

(d) $\left\{\frac{2\pi}{3}\right\}$

14. Let $\sin\alpha + \cos\beta = \frac{2}{3}, \cos\alpha + \sin\beta = \frac{1}{3}$, where $0 < \alpha, \beta < \frac{\pi}{2}$ then

$$\tan^{-1}\left(\frac{\sqrt{1+\sin(\alpha-\beta)} - \sqrt{1-\sin(\alpha-\beta)}}{\sqrt{1+\sin(\alpha-\beta)} + \sqrt{1-\sin(\alpha-\beta)}}\right) =$$

(a) $\cot^{-1}\left(\frac{1}{3}\right)$ (b) $\tan^{-1}\left(\frac{1}{3}\right)$

(c) $\cot^{-1}\left(\frac{2}{5}\right)$ (d) $\tan^{-1}\left(\frac{2}{5}\right)$

15. The number of solutions of the equation $\sec^{-1}\left(\frac{4\sqrt{3}x}{x^2+3}\right) + \tan^{-1}\sqrt{6x^2 - x^4 - 9} = \pi/3$ is

(a) 0 (b) 3 (c) 2 (d) 1

16. In a ΔABC, $\sin(A-B) = \frac{3}{5}$, $\sin A = 2\sin B$, then $\frac{a+b}{c} =$

(a) $\frac{4}{\sqrt{5}}$ (b) $\sqrt{\frac{2}{5}}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{5}}$

17. If $\lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{2r^2}\right) = \alpha$, then $\cot\frac{\alpha}{2} =$

(a) 1 (b) $\sqrt{2}-1$ (c) $\sqrt{2}+1$ (d) $\frac{\sqrt{5}-1}{2}$

18. *ABCD* is a trapezium such that *AB*, *DC* are parallel and *BC* is perpendicular to them. If $\angle ADB = \frac{\pi}{4}, BC = 3, CD = 4$, then $AB =$

(a) $\frac{14}{3}$ (b) $\frac{7}{\sqrt{3}}$ (c) $\frac{13}{4}$ (d) $\frac{25}{7}$

19. The value of

$$\tan^{-1}\left(\cos\left(2\tan^{-1}\frac{3}{4}\right) + \sin\left(2\cot^{-1}\frac{1}{2}\right)\right) \text{ is}$$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $> \frac{\pi}{4}$ (d) $< \frac{\pi}{4}$

20. Let $x \in \left(\pi, \frac{3\pi}{2}\right)$, then $|\cos^{-1}(\cos x)| + |\tan^{-1}(\tan x)|$ is equal to

(a) $2x - 3\pi$ (b) $-\pi$ (c) $2x - \pi$ (d) π

21. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$ and $\operatorname{Arg} z - \operatorname{Arg} \omega = \pi/2$, then $\bar{z}\omega =$

(a) 1 (b) -1 (c) i (d) $-i$

22. Let z and ω be complex numbers such that $\bar{z} + i\bar{\omega} = 0$ and $\arg z\omega = \pi$, then $\arg z =$

(a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) $5\pi/4$

23. If the imaginary part of the expression $\frac{z-1}{e^{i\theta}} + \frac{e^{i\theta}}{z-1}$ be zero, then locus of z is

(a) straight line (b) parabola

(c) unit circle (d) ellipse

24. If ω is a complex number such that $|\omega| = r \neq 1$ then $z = \omega + \frac{1}{\omega}$ describes a conic. The distance between the foci is

(a) 2 (b) $2(\sqrt{2}-1)$

(c) 3 (d) 4

25. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(a) a line not passing through the origin

(b) $|z| = \sqrt{2}$ (c) the x-axis

(d) the y-axis

26. The number of solutions of the system of equations given by $|z| = 3$ and $|z+1-i| = \sqrt{2}$ is equal to

(a) 4 (b) 2

(c) 1 (d) no solution

27. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \operatorname{Im}(z^{2m-1})$ at $\theta = 2°$ is

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$ (c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

28. The roots of $1 + z + z^3 + z^4 = 0$ are represented by the vertices of

(a) a square (b) an equilateral triangle

(c) a rhombus (d) a rectangle

29. If z_1, z_2 and z_3 be the vertices of ΔABC, taken in anti-clock wise direction and z_0 be the circumcentre, then $\left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

30. If a, b, c, a_1, b_1, c_1 are non-zero complex numbers satisfying $\frac{a}{a_1}+\frac{b}{b_1}+\frac{c}{c_1}=1+i$ and $\frac{a_1}{a}+\frac{b_1}{b}+\frac{c_1}{c}=0$, then $\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}$ is equal to

(a) $2i$ (b) $2+2i$ (c) 2 (d) $2-2i$

31. Let two non collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time 't', the position vector $\overline{OP}$ (where O is the origin) is given by $\hat{a}\cos t+\hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overline{OP}$ and $\hat{u}$ be the unit vector along $\overline{OP}$ then,

(a) $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

(b) $\hat{u}=\frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

(c) $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

(d) $\hat{u}=\frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

32. Let $\vec{a}=2\hat{i}+3\hat{j}$ and $\hat{b}$ be any vector in xy-plane perpendicular to $\vec{a}$. If a vector $\vec{c}$ in the same plane have lengths of projections along $\vec{a}$ and $\vec{b}$ equal to $\frac{5}{\sqrt{13}}$ and $\frac{3}{\sqrt{13}}$, then $\vec{c}=$

(a) $\frac{2}{13}\hat{i}+\frac{3}{13}\hat{j}$ (b) $\frac{9}{13}\hat{i}+\frac{19}{13}\hat{j}$

(c) $\frac{19}{13}\hat{i}+\frac{9}{13}\hat{j}$ (d) $\frac{19}{13}\hat{i}+\frac{8}{13}\hat{j}$

33. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors equally inclined to each other at an angle α ($\neq 0$) then the angle between $\vec{a}\times\vec{b}$ and the plane containing $\vec{b}$ and $\vec{c}$ is

(a) $\sin^{-1}\left(\tan\frac{\alpha}{2}.|\cot\alpha|\right)$

(b) $\cos^{-1}\left(\tan\frac{\alpha}{2}.|\cot\alpha|\right)$

(c) $\cos^{-1}\left(\cot\frac{\alpha}{2}.|\tan\alpha|\right)$

(d) $\sin^{-1}\left(\cot\frac{\alpha}{2}.|\tan\alpha|\right)$

34. Let $A(\vec{a})$ and $B(\vec{b})$ be points on two skew lines $\vec{r}=\vec{a}+\lambda\vec{p}$ and $\vec{r}=\vec{b}+\mu\vec{q}$ and the shortest distance between the skew lines is 1, where $\vec{p}$ and $\vec{q}$ are unit vectors forming adjacent sides of a parallelogram enclosing an area of 1/2 units. If an angle between AB and the line of shortest distance is 60°, then $AB=$

(a) 1/2 (b) 2

(c) 1 (d) $\lambda \in R-\{0\}$

35. If $\vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b}=\vec{c}\cdot\vec{c}=1$; $\vec{a}\cdot\vec{b}=\frac{1}{2}$; $\vec{b}\cdot\vec{c}=\frac{1}{\sqrt{2}}$; $\vec{c}\cdot\vec{a}=\frac{\sqrt{3}}{2}$ then $[\vec{a}\ \vec{b}\ \vec{c}]$ is

(a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(c) $\frac{\sqrt{\sqrt{6}+2}}{2}$ (d) $\frac{\sqrt{\sqrt{6}-2}}{2}$

36. If $\vec{u}=\vec{a}-\vec{b}$, $\vec{v}=\vec{a}+\vec{b}$ and $|\vec{a}|=|\vec{b}|=2$, then $|\vec{u}\times\vec{v}|=$

(a) $2\sqrt{[16-(\vec{a}\cdot\vec{b})^2]}$ (b) $\sqrt{[16-(\vec{a}\cdot\vec{b})^2]}$

(c) $2\sqrt{[4-(\vec{a}\cdot\vec{b})^2]}$ (d) $\sqrt{[4-(\vec{a}\cdot\vec{b})^2]}$

37. Let the vectors, $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$. Let P_1 and P_2 be planes determined by the pair of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2 is

(a) 0 (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

SECTION-II

(Multiple Correct Answer(s) Type)

This section contains multiple correct answer(s) type questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

38. If $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b}, a,b,c>0$ then

(a) $a^{b+c}\cdot b^{c+a}\cdot c^{a+b}=1$

(b) $a^{b+c}+b^{c+a}+c^{a+b}\geq 3$

(c) $a^{b+c}b^{c+a}c^{a+b}=3$

(d) $a^{b+c}+b^{c+a}+c^{a+b}\geq 3(3^{1/3})$

39. If $x=\sec\theta-\tan\theta$ and $y=\operatorname{cosec}\theta+\cot\theta$, then

(a) $x=\frac{y+1}{y-1}$ (b) $x=\frac{y-1}{y+1}$

(c) $y=\frac{1+x}{1-x}$ (d) $xy+x-y+1=0$

40. $\sin\theta+\sqrt{3}\cos\theta=6x-x^2-11,\ 0\le\theta\le4\pi,\ x\in R$ holds for
(a) no values of x and θ
(b) one value of x and two values of θ
(c) two values of x and two values of θ
(d) two pairs of values of (x, θ)

41. Let $x\in\left(\frac{5\pi}{2},3\pi\right)$ then
(a) $\cos^{-1}(\sin(\cos^{-1}(\cos x)-2\tan^{-1}(\tan x)))=\frac{7\pi}{2}-x$
(b) $\cos^{-1}(\sin(\cos^{-1}(\cos x)-2\tan^{-1}(\tan x)))=\frac{9\pi}{2}-x$
(c) $\tan^{-1}(\cot(2\sin^{-1}(\sin x)-\tan^{-1}(\tan x)))=\frac{\pi}{2}+x$
(d) $\tan^{-1}(\cot(2\sin^{-1}(\sin x)+\tan^{-1}(\tan x)))=x-\frac{5\pi}{2}$

42. In a triangle ABC, if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to
(a) 45° (b) 135° (c) 120° (d) 60°

43. Let ABC be an isosceles triangle with base BC. If 'r' is the radius of the circle inscribed in the ΔABC and r_1 be the radius of the circle described opposite to the angle A, then $r_1r =$
(a) $R^2\sin^2A$ (b) $R^2\sin^22B$
(c) $\frac{1}{2}a^2$ (d) $\frac{a^2}{4}$

44. If A is the area and $2s$ the sum of the sides of a triangle, then
(a) $A<\frac{s^2}{4}$ (b) $A\le\frac{s^2}{3\sqrt{3}}$
(c) $A>\frac{s^2}{\sqrt{3}}$ (d) $A\le\frac{s^2}{\sqrt{3}}$

45. If $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^2$, then $D=\begin{vmatrix}x^{N_1} & y^{N_2}\\ z^{N_3} & w^{N_4}\end{vmatrix}(N_1,N_2,N_3,N_4\in N)$
(a) has maximum value of 2
(b) has minimum value of 0
(c) has maximum value 1
(d) has minimum value of -2

46. Which of the following quantities is/are positive ?
(a) $\cos(\tan^{-1}(\tan4))$ (b) $\sin(\cot^{-1}(\cot4))$
(c) $\tan(\cos^{-1}(\cos5))$ (d) $\cot(\sin^{-1}(\sin4))$

47. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $Re(z_1z_2) = 0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies
(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$
(c) $\text{Re}(\omega_1\,\omega_2) = 0$ (d) $|\omega_1| = 2$

48. If $z_1 = 5 + 12i$ and $|z_2| = 4$, then
(a) maximum $(|z_1 + iz_2|) = 17$
(b) minimum $(|z_1 + (1 + i)z_2|) = 13-9\sqrt{2}$
(c) minimum $\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{4}$
(d) maximum $\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{3}$

49. If z is a complex number satisfying $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$, then z lies on
(a) $y = x$ (b) $y = -x$
(c) $y = x + 1$ (d) $y = -x + 1$

50. A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane and the circumcentre of the triangle ABC lies at the origin. If the altitude AD of the triangle ABC meets the circumcircle again at P, then P represents the complex number
(a) $-\bar{z}_1z_2z_3$ (b) $-\frac{\bar{z}_1z_2}{\bar{z}_3}$
(c) $-\frac{\bar{z}_1z_3}{\bar{z}_2}$ (d) $-\frac{z_2z_3}{z_1}$

51. If points A and B are represented by the non-zero complex numbers z_1 and z_2 on the Argand plane such that $|z_1 + z_2| = |z_1 - z_2|$ and O is the origin, then
(a) orthocentre of ΔOAB lies at O
(b) circumcentre of ΔAOB is $\frac{z_1+z_2}{2}$
(c) $\arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}$
(d) ΔOAB is isosceles

52. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $h(x) = x\,f(x^3) + x^2g(x^6)$ is divisible by $x^2 + x + 1$, then
(a) $f(1) = g(1)$ (b) $f(1) = -g(1)$
(c) $h(1) = 0$ (d) none of these

53. If α $(\alpha\ne1)$ is the fifth root of unity then
(a) $|1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4| = 0$
(b) $|1 + \alpha + \alpha^2 + \alpha^3| = 1$

(c) $|1+\alpha+\alpha^2| = 2\cos\frac{\pi}{5}$

(d) $|1+\alpha| = 2\cos\frac{\pi}{10}$

54. If the lines $a\bar{z}+\bar{a}z+b=0$ and $c\bar{z}+\bar{c}z+d=0$ are mutually perpendicular, where a and c are non-zero complex numbers and b and d are real numbers, then

(a) $a\bar{a}+c\bar{c}=0$ (b) $a\bar{c}$ is purely imaginary

(c) $\arg\left(\frac{a}{c}\right)=\pm\frac{\pi}{2}$ (d) $\frac{a}{\bar{a}}=\frac{c}{\bar{c}}$

55. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}|=4$ with $\vec{a}\times\vec{b}=2\vec{a}\times\vec{c}$. The angle between $\vec{a}$ and $\vec{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. Then $\vec{b}-2\vec{c}=\lambda\vec{a}$, if λ is

(a) 3 (b) 1/4 (c) –4 (d) –1/4

56. A parallelogram is constructed on the vectors $\vec{a}=3\vec{\alpha}-\vec{\beta}$, $\vec{b}=\vec{\alpha}+3\vec{\beta}$. If $|\vec{\alpha}|=|\vec{\beta}|=2$ and angle between $\vec{\alpha}$ and $\vec{\beta}$ is $\pi/3$, then the length of diagonal(s) of the parallelogram is

(a) $4\sqrt{5}$ (b) $4\sqrt{3}$ (c) $4\sqrt{7}$ (d) $4\sqrt{2}$

SECTION-III

(Comprehension Type)

This section contains paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

Paragraph for Question Nos. 57 to 59

Trigonometric equations which involve secant and cosecant are generally solved by converting them into their reciprocals cosine and sine. Care must be taken to avoid invalid solutions. Answer the following questions for the trigonometric equation $\sec\theta + \operatorname{cosec}\theta = c$ which is defined for $\theta \neq K\frac{\pi}{2}$, where $K \in I$

57. If $c=\sqrt{8}$ and $\theta \in (0, 2\pi)$, the no. of solutions of the equation is

(a) 1 (b) 2 (c) 4 (d) 3

58. If $c^2 < 8$ and $\theta \in (0, 2\pi)$, the no. of solutions of the equation is

(a) 1 (b) 2 (c) 3 (d) 4

59. If $c^2 > 8$ and $\theta \in (0, 2\pi)$, the no. of solutions of the equation is

(a) 2 (b) 4 (c) 8 (d) 6

Paragraph for Question Nos. 60 to 62

An equation of the form $2m\log_a f(x) = \log_a g(x)$, $a > 0$, $a \neq 1$, $m \in N$ is equivalent to the system

$$\begin{cases} f(x) > 0 \\ f_{(x)}^{2m} = g(x) \end{cases}$$

60. The no. of solutions of $2\log 2x = \log_e(7x - 2 - 2x^2)$ is

(a) 1 (b) 2

(c) 3 (d) Infinite

61. The solution set of the equation $\log_{(x^3+6)}(x^2-1) = \log_{(2x^2+5x)}(x^2-1)$ is

(a) $\{-2\}$ (b) $\{1\}$ (c) $\{3\}$ (d) $\{-2, 1, 3\}$

62. The solution set of the equation $\log_{10}(x-9)+2\log_{10}\sqrt{2x-1}=2$ is

(a) $\{\phi\}$ (b) $\{1\}$ (c) $\{2\}$ (d) $\{13\}$

Paragraph for Question Nos. 63 to 65

If $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ are the roots of the equation $8x^3 - 4x^2 - 4x + 1 = 0$

63. The value of $\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}$ is

(a) 2 (b) 4 (c) 8 (d) 9

64. The equation whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$ is

(a) $x^3 - 35x^2 + 7x - 21 = 0$

(b) $x^3 - 35x^2 + 21x - 7 = 0$

(c) $x^3 - 21x^2 + 35x - 7 = 0$

(d) $x^3 - 21x^2 + 7x - 35 = 0$

65. The value of $\left(\sum_{r=1}^{3}\tan^2\left(\frac{2r-1}{7}\right)\right)\left(\sum_{r=1}^{3}\cot^2\left(\frac{2r-1}{7}\right)\right)$ is

(a) 15 (b) 105 (c) 21 (d) 147

Paragraph for Question Nos. 66 to 68

To solve a trigonometric inequation of the type $\sin x \geq a$ where $|a| \leq 1$, we take a hill of length 2π in the sine curve and write the solution within that hill. For the general solution, we add $2n\pi$. For instance, to solve $\sin x \geq -\frac{1}{2}$, we take the hill $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ over which the solution is $-\frac{\pi}{6} < x < \frac{7\pi}{6}$ and the general solution is $2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}, n \in I$. The inequations

$\cos x \geq a$, $\cos x \leq a$ where $|a| \leq 1$, are solved similarly. To solve a trigonometric inequation of general nature we bring it to the canonical form. *i.e.*, in one of the forms $\sin x \geq a$, $\sin x \leq a$, $\cos x \geq a$ or $\cos x \leq a$.

66. Solution to the inequation $\sin^6 x + \cos^6 x < \frac{7}{16}$ is

(a) $n\pi + \frac{\pi}{3} < x < n\pi + \frac{\pi}{2}$

(b) $2n\pi + \frac{\pi}{3} < x < 2n\pi + \frac{\pi}{2}$

(c) $\frac{n\pi}{2} + \frac{\pi}{6} < x < \frac{n\pi}{2} + \frac{\pi}{3}$

(d) $2n\pi - \frac{\pi}{3} < x < 2n\pi + \frac{\pi}{2}$

67. Solution to the inequality $\cos 2x + 5\cos x + 3 \geq 0$, over $[-\pi, \pi]$ is

(a) $[-\pi, \pi]$ (b) $\left[-\frac{5\pi}{6}, \frac{5\pi}{6}\right]$

(c) $[0, \pi]$ (d) $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$

68. The solution of $2\sin^2\left(x + \frac{\pi}{4}\right) + \sqrt{3}\cos 2x \geq 0$ over $[-\pi, \pi]$ is

(a) $[-\pi, \pi]$ (b) $\left[-\frac{5\pi}{6}, \frac{5\pi}{6}\right]$

(c) $[0, \pi]$ (d) $\left[-\pi, -\frac{7\pi}{6}\right] \cup \left[\frac{3\pi}{4}, \frac{17\pi}{12}\right]$

Paragraph for Question Nos. 69 to 71

Consider the following definitions of inverse trigonometric functions.

$f(x)$	Domain	Co-Domain
$\tan^{-1} x$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	R
$\cot^{-1} x$	$(\pi, 2\pi)$	R

and $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \; \forall \, x \in [-1, 1]$

69. $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) =$

(a) $\frac{\pi}{4}$ (b) $\frac{11\pi}{4}$ (c) $\frac{13\pi}{4}$ (d) $\frac{9\pi}{4}$

70. $2\tan^{-1}\left(\frac{1}{3}\right) + \cot^{-1}(7) =$

(a) $\frac{\pi}{4}$ (b) $\frac{9\pi}{4}$ (c) $\frac{17\pi}{4}$ (d) $\frac{13\pi}{4}$

71. $\cos\left(\tan^{-1}(\sin(\cot^{-1}\sqrt{2}))\right) =$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{-\sqrt{3}}{2}$ (c) $\frac{-1}{2}$ (d) $\frac{1}{2}$

SECTION-IV

(Matrix-Match Type)

This section contains questions. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example.

If the correct matches are A – p, s, B – q, r, C – p, q and D – s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
A	●	○	○	●
B	○	●	●	○
C	●	●	○	○
D	○	○	○	●

72. Match the following:

	Column I		Column II
(A)	Range of $\sin^{-1}3x + \cos^{-1}3x + \tan^{-1}3x$ contains	(p)	0
(B)	Range of $\cot^{-1}(\sqrt{2x - 1 - x^2}) + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ contains	(q)	π
(C)	Range of $\sin^{-1}\left(\left[x + \frac{7}{4}\right]\right) + \cos^{-1}\left(\left[x + \frac{15}{4}\right]\right) + \frac{\pi}{2}$ contains (where $[\cdot]$ denotes greatest integer function)	(r)	$\pi/4$
(D)	Range of $\tan^{-1}(\sqrt{x^2 + 3x}) + \cos^{-1}(\sqrt{x^2 + 3x + 1})$ contains	(s)	$\pi/2$

73. Let $f(x) = \cos^{-1}\left(\frac{x - \sqrt{1 - x^2}}{\sqrt{2}}\right)$, $g(x) = \cos^{-1} x$

	Column I		Column II
(A)	$f(\sqrt{2} - 1) - g(\sqrt{2} - 1) =$	(p)	$\pi/4$
(B)	$f\left(-\frac{4}{5}\right) + g\left(-\frac{4}{5}\right) =$	(q)	$7\pi/4$
(C)	$g(\sqrt{3} - 1) - f(\sqrt{3} - 1) =$	(r)	$-\pi/4$

(D)	$g\left(\frac{-9}{10}\right)+f\left(\frac{-9}{10}\right)+\frac{\pi}{4}=$	(s)	2π

74. Number of solutions of

Column I		Column II	
(A)	$z^2 + \|z\| = 0$	(p)	1
(B)	$z^2+\bar{z}^2=0$	(q)	3
(C)	$z^2+8\bar{z}=0$	(r)	4
(D)	$\|z-2\| = 1$ and $\|z-1\| = 2$	(s)	infinite

SECTION-V

(Integer Answer Type Questions)

This section contains questions. The answer to each of the questions is a single digit integer, ranging from 0 to 9.

75. If $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0$ where a, b, c are three distinct complex numbers, then the value of $\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}$ is equal to

76. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then $|z_1 + z_2 + z_3|$ is equal to

77. If the complex numbers z for which $\arg\left[\frac{3z-6-3i}{2z-8-6i}\right]=\frac{\pi}{4}$ and $|z - 3 + i| = 3$, are $\left(k-\frac{4}{\sqrt{5}}\right)+i\left(1+\frac{2}{\sqrt{5}}\right)$ and $\left(k+\frac{4}{\sqrt{5}}\right)+i\left(1-\frac{2}{\sqrt{5}}\right)$, then k must be equal to

78. If $\alpha = e^{2\pi i/7}$ and $f(x)=A_0+\sum_{k=1}^{20}A_kx^k$, then the value of $f(x) + f(\alpha x) + f(\alpha^2x) + \ldots + f(\alpha^6x)$ is $k(A_0 + A_7x^7 + A_{14}x^{14})$, then k must be equal to

79. If magnitude of a complex number $4 - 3i$ is tripled and rotated by an angle π anticlockwise about origin then resulting complex number would $-12 + \lambda i$, then λ must be equal to

80. The maximum value of $|z|$ when z satisfies the condition $\left|z-\frac{2}{z}\right|=2$ is $\sqrt{\lambda}+1$

81. z_1, z_2 are roots of the equation $z^2 + az + b = 0$. If O is origin, A and B represent z_1 and z_2 and ΔAOB is equilateral, then $\frac{a^2}{b}$ is equal to ……..

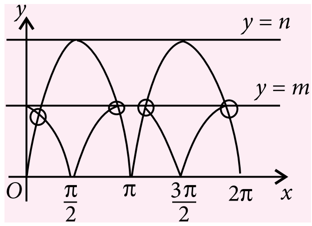
SOLUTIONS

1. (a) : $2\cos 2x - 1 = 2(\cos^2x - \sin^2x) - (\cos^2x + \sin^2x)$
$= \cos^2x - 3\sin^2x$

$\sin x=\frac{b}{b-1}$

$-1\le\frac{b}{b-1}\le 1$

2. (d) : The graph shows the number of solutions



3. (a)

4. (d) : $\sin x+\sin\left(\frac{\pi}{2}-y\right)=a$

$2\sin\frac{x+\frac{\pi}{2}-y}{2}\cos\frac{x-\frac{\pi}{2}+y}{2}=a$

$\cos x+\cos\left(\frac{\pi}{2}-y\right)=b$

$\Rightarrow\ 2\cos\frac{x+\frac{\pi}{2}-y}{2}\cos\frac{x-\frac{\pi}{2}+y}{2}=b$

$\Rightarrow\ \tan\left(\frac{\pi}{4}+\frac{x-y}{2}\right)=\frac{a}{b}$

5. (d) : $\frac{a+b}{3}=\sqrt{ab}\Rightarrow(a+b)^2=9ab\Rightarrow\frac{a}{b}+\frac{b}{a}=7$

6. (a) : $\log_{10}(\tan 1°\ \tan 2°\ \ldots\ \tan 89°) = \log_{10}1 = 0$

7. (b) : $(81)^{\frac{1}{\log_5 9}}=5^2=25$

$3^{\left(\frac{3}{\log_{\sqrt{6}}3}\right)}=(\sqrt{6})^3=6\sqrt{6}$

$(\sqrt{7})^{\frac{2}{\log_{25}7}}=25$

$(125)^{\log_{25}6}=\sqrt{216}=6\sqrt{6}$

8. (a) : $x=\log_{12}6=\frac{\log 2+\log 3}{2\log 2+\log 3}$

$$\frac{\log 3}{\log 2} = \frac{1-2x}{x-1}$$

$$y = \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2} \Rightarrow \frac{\log 3}{\log 2} = \frac{1-3y}{y-3}$$

9. (b)

10. (a) : $32\sin\frac{A}{2}\sin\frac{5A}{2} = 16(\cos 2A - \cos 3A)$

$$\cos 2A = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{18}{16} - 1 = \frac{2}{16}$$

$$\cos 3A = 4\left(\frac{3}{4}\right)^3 - 3\left(\frac{3}{4}\right) = \frac{27}{16} - \frac{36}{16} = -\frac{9}{16}$$

11. (a) : $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}, \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$$\frac{\sin 15^\circ}{\sin x} + \frac{\cos 15^\circ}{\cos x} = 2$$

$\sin(15^\circ + x) = \sin 2x = \sin(180^\circ - 2x)$

$\Rightarrow x = 55^\circ = \frac{11\pi}{36} \Rightarrow \lambda = 11$

12. (c)

13. (a) : $\sin x \le \frac{1}{2}$ or $\sin x = 1$

14. (b) : $\frac{\sin\alpha + \cos\beta}{\cos\alpha + \sin\beta} = \tan\left(\frac{\pi}{4} + \frac{\alpha-\beta}{2}\right) = 2$

$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{3}$

$\Rightarrow 0 < \frac{\alpha-\beta}{2} < \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\tan\left(\frac{\alpha-\beta}{2}\right)\right) = \frac{\alpha-\beta}{2}$

15. (d) : $-9 + 6x^2 - x^4 = -(x^3 - 3)^2$

$\Rightarrow \tan^{-1}$ will be defined only for $x = \pm\sqrt{3}$.

For $x = \sqrt{3}$, it is $\frac{\pi}{3}$.

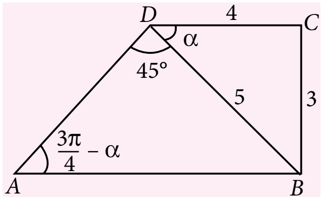
16. (c) : $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3} = \frac{a-b}{a+b}\cot\frac{C}{2} \Rightarrow \angle C = 90^\circ$

$$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}} = \frac{\frac{3}{\sqrt{10}}}{\frac{1}{\sqrt{2}}} = \frac{3}{\sqrt{5}}$$

17. (c) : $\lim_{n\to\infty} \tan^{-1}(2n+1) - \tan^{-1}(1) = \frac{\pi}{4}$

$\Rightarrow \cot\frac{\pi}{8} = \sqrt{2} + 1$

18. (d) : $\frac{AB}{\sin 45^\circ} = \frac{5}{\sin\left(\frac{3\pi}{4} - \alpha\right)}$



$\Rightarrow AB = \frac{25}{7}$ m

19. (c) : $\sin\left(2\cot^{-1}\frac{1}{2}\right) = \frac{4}{5}$

$$\cos\left(2\tan^{-1}\frac{3}{4}\right) = \frac{7}{25}$$

20. (d) : $\cos^{-1}(\cos x) = -x + 2\pi$, $\tan^{-1}(\tan x) = -x + \pi$

21. (d) : $|z\omega| = 1 \Rightarrow |z||\omega| = 1$

$\therefore |\omega| = \frac{1}{|z|}$ ... (1)

Let $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$

Given, Arg ω = Arg $z - \pi/2 = \theta - \pi/2$

$\therefore \omega = \frac{1}{r}e^{i(\theta-\pi/2)}$ by (1)

$\therefore \bar{z}\omega = (re^{-i\theta})\cdot\frac{1}{r}e^{i(\theta-\pi/2)} = e^{-i\pi/2}$

$$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$$

22. (c) : Let $z = re^{i\theta}$. Given, $\arg z + \arg\omega = \pi$

$\therefore \arg\omega = \pi - \theta \quad \therefore \omega = r_1 e^{i(\pi-\theta)}$

Now $\bar{z} + i\bar{\omega} = 0 \Rightarrow re^{-i\theta} + ir_1 e^{-i(\pi-\theta)} = 0$

or, $r(\cos\theta - i\sin\theta) + ir_1[\cos(\pi - \theta) - i\sin(\pi - \theta)] = 0$

or, $r(\cos\theta - i\sin\theta) + r_1(-i\cos\theta + \sin\theta) = 0$

$\therefore$ Equating real and imaginary parts

$r\cos\theta + r_1\sin\theta = 0$

$-r\sin\theta - r_1\cos\theta = 0$

$\therefore \frac{r}{r_1} = -\tan\theta = \frac{r_1}{r}$ or $r^2 = r_1^2 \quad \therefore r = r_1$

$\therefore \tan\theta = -1$ or $\theta = \arg z = \frac{3\pi}{4}$

23. (c) : Given I.P. of $U + \frac{1}{U} = 0$ where $U = \frac{z-1}{e^{i\theta}}$

Now, I.P. of $Z = 0 \Rightarrow Z - \bar{Z} = 2iY = 0$

$\therefore \left(U + \frac{1}{U}\right) - \overline{\left(U + \frac{1}{U}\right)} = 0$

$$\text{or, } \left(U+\frac{1}{U}\right)-\left(\bar{U}+\frac{1}{\bar{U}}\right)=0$$

$$\text{or, } (U-\bar{U})+\left(\frac{1}{U}-\frac{1}{\bar{U}}\right)=0$$

$$\text{or, } (U-\bar{U})-\frac{U-\bar{U}}{U\bar{U}}=0$$

$$\text{or, } (U-\bar{U})\left(1-\frac{1}{|U|^2}\right)=0$$

or, $|U|^2 = 1$ as $U \neq \bar{U}$

$$\Rightarrow \left|\frac{z-1}{e^{i\theta}}\right|^2=\frac{|z-1|^2}{1}=1 \text{ as } |e^{i\theta}|=1$$

$\Rightarrow$ $|z - 1| = 1$, which represents a unit circle.

24. (d) : If $\omega = re^{i\theta}$, then $\frac{1}{\omega}=\frac{1}{r}e^{-i\theta}$

$$\therefore\ z=\left(\omega+\frac{1}{\omega}\right)$$

$$=r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)$$

$$\therefore\ x=\left(r+\frac{1}{r}\right)\cos\theta,\ y=\left(r-\frac{1}{r}\right)\sin\theta$$

Eliminating θ, $\dfrac{x^2}{\left(r+\frac{1}{r}\right)^2}+\dfrac{y^2}{\left(r-\frac{1}{r}\right)^2}=1$

Above represents an ellipse and distance between foci is

$$2ae=2\sqrt{a^2\left(1-\frac{b^2}{a^2}\right)}=2\sqrt{a^2-b^2}=2\sqrt{4}=4$$

25. (d) : Since $|z| = 1$ $\therefore$ $z = e^{i\theta}$ as $r = 1$

$$\therefore\ \frac{z}{1-z^2}=\frac{1}{z^{-1}-z}=\frac{1}{e^{-i\theta}-e^{i\theta}}=\frac{-1}{2i\sin\theta}=0+i\frac{1}{2\sin\theta}$$

Real part of $\dfrac{z}{1-z^2}$ is zero. Hence it lies on y-axis.

26. (d) : The two equations represent circles $x^2 + y^2 = 9$, $(x + 1)^2 + (y - 1)^2 = 2$

$C_1(0,0)$, $r_1 = 3$ and $C_2(-1,1)$, $r_2=\sqrt{2}$

$C_1C_2=\sqrt{2}$; $r_1-r_2=3-\sqrt{2}$

Now $\sqrt{2}<3-\sqrt{2}$ as $2\sqrt{2}<3 \Rightarrow 8 < 9$

$\therefore$ $C_1C_2 < r_1 - r_2$. Hence the two circles do not intersect but one lies completely within the other. Hence there is no solution.

27. (d) : $z = \cos\theta + i\sin\theta$

$\therefore$ $z^m = \cos m\theta + i \sin m\theta$

$\therefore$ Im $(z^{2m-1}) = \sin(2m - 1)\theta$

$$\therefore\ E=\sum_{m=1}^{15}\sin(2m-1)\theta$$

$= \sin\theta + \sin3\theta + \sin5\theta + \ldots.. + \sin29\theta$

$$=\frac{\sin m.\frac{2\theta}{2}}{\sin\frac{2\theta}{2}}\sin\left[\frac{\theta+29\theta}{2}\right]$$

$$=\frac{\sin 15\theta}{\sin\theta}\sin 15\theta=\frac{\sin 30^\circ \sin 30^\circ}{\sin 2^\circ} \quad (\because m=15 \text{ and } \theta=2^\circ)$$

$$=\frac{1}{4\sin 2^\circ}$$

28. (b) : The given equation is $(1 + z)(1 + z^3) = 0$. The distinct roots being -1, $-\omega$, $-\omega^2$, which are represented by points a, b and c in that order

$ab = |1 - \omega| = |\omega|\,|\omega^2 - 1| = |\omega^2 - 1|$

$bc = |\omega - \omega^2| = |\omega^2|\,|\omega^2 - 1| = |\omega^2 - 1|$

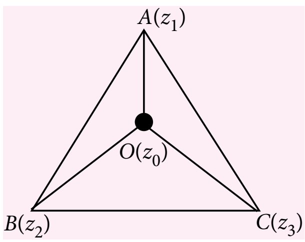
$ca = |\omega^2 - 1|$

The three points represent the vertices of an equilateral triangle.

29. (c) : Taking rotation at O

$$\frac{z_0-z_1}{z_0-z_2}=\cos 2C-i\sin 2C$$

$$\frac{z_0-z_3}{z_0-z_2}=\cos 2A+i\sin 2A$$



$$\text{Now } \left(\frac{z_0-z_1}{z_0-z_2}\right)\frac{\sin 2A}{\sin 2B}+\left(\frac{z_0-z_3}{z_0-z_2}\right)\frac{\sin 2C}{\sin 2B}$$

$$=\frac{\sin 2A\cos 2C-i\sin 2A\sin 2C+\cos 2A\sin 2C+i\sin 2A\sin 2C}{\sin 2B}$$

$$=\frac{\sin(2A+2C)}{\sin 2B}=-1$$

30. (a) : $\dfrac{a}{a_1}+\dfrac{b}{b_1}+\dfrac{c}{c_1}=1+i$

$$\Rightarrow \frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}=2i-2\left(\frac{ab}{a_1b_1}+\frac{bc}{b_1c_1}+\frac{ac}{a_1c_1}\right)$$

$$=2i-\frac{2abc}{a_1b_1c_1}\left(\frac{c_1}{c}+\frac{b_1}{b}+\frac{a_1}{a}\right)=\ 2i-0=2i$$

31. (a) : $M=\sqrt{(\cos^2 t+\sin^2 t+2\hat{a}\cdot\hat{b}\sin t\cos t)}$

$$=\sqrt{1+\hat{a}\cdot\hat{b}}$$

$\sin 2t$ is maximum at $t = \pi/4$

$$\hat{u}=\frac{\hat{a}\frac{1}{\sqrt{2}}+\hat{b}\frac{1}{\sqrt{2}}}{\left|\hat{a}\frac{1}{\sqrt{2}}+\hat{b}\frac{1}{\sqrt{2}}\right|}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

32. (c) : Let $\vec{b}=x\hat{i}+y\hat{j}$. Since, $\vec{b}$ is perpendicular to $\vec{a}$, so $2x + 3y = 0$

$\Rightarrow\ \vec{b}=x\left(\hat{i}-\frac{2}{3}\hat{j}\right)$. Let $\vec{c}=c_1\hat{i}+c_2\hat{j}$

Projection of $\vec{c}$ along $\vec{a}$ $=\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|}=\frac{2c_1+3c_2}{\sqrt{13}}=\frac{5}{\sqrt{13}}$

$\Rightarrow\ 2c_1 + 3c_2 = 5$ ... (1)

Also, projection of $\vec{c}$ along $\vec{b}=\frac{\vec{c}\cdot\vec{b}}{|\vec{b}|}$

$$=\frac{x\left(c_1-\frac{2c_2}{3}\right)}{x\sqrt{1+\frac{4}{9}}}=\frac{3}{\sqrt{13}}$$

$\Rightarrow\ c_1-\frac{2}{3}c_2=1$ .....(2)

Solving equations (1) & (2), we get

$$c_2=\frac{9}{13},\ c_1=\frac{19}{13}$$

33. (a) : Let θ be the required angle, then

$$\sin\theta=\cos(90°-\theta)=\left|\frac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})}{|\vec{a}\times\vec{b}||\vec{b}\times\vec{c}|}\right|$$

$$=\left|\frac{[(\vec{a}\times\vec{b})\times\vec{b}]\cdot\vec{c}}{\sin^2\alpha}\right|=\left|\frac{[(\vec{a}\cdot\vec{b})\vec{b}-(\vec{b}\cdot\vec{b})\vec{a}]\cdot\vec{c}}{\sin^2\alpha}\right|$$

$$=\left|\frac{\cos\alpha\,(\vec{b}\cdot\vec{c})-(\vec{a}\cdot\vec{c})}{\sin^2\alpha}\right|=\left|\frac{\cos^2\alpha-\cos\alpha}{\sin^2\alpha}\right|$$

$$=\left|-\frac{\cos\alpha(1-\cos\alpha)}{\sin\alpha\sin\alpha}\right|=\left|-\frac{\cos\alpha\cdot 2\sin^2(\alpha/2)}{\sin\alpha\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right|$$

$$=\left|-\cot\alpha\tan\frac{\alpha}{2}\right|=\left|\cot\alpha\tan\frac{\alpha}{2}\right|$$

$$\therefore\ \theta=\sin^{-1}\left(\tan\frac{\alpha}{2}|\cot\alpha|\right)$$

34. (b) : $I=\left|(\vec{b}-\vec{a})\cdot\frac{(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|\Rightarrow|\vec{b}-\vec{a}||\vec{p}\times\vec{q}|\cos 60°=\frac{1}{2}$

$$\Rightarrow\ AB\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}\ \Rightarrow\ AB=2$$

35. (d) : $[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}$

$$=\begin{vmatrix}1 & \frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{1}{2} & 1 & \frac{1}{\sqrt{2}}\\ \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} & 1\end{vmatrix}$$

$$=1\left(1-\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2\sqrt{2}}\right)+\frac{\sqrt{3}}{2}\left(\frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2}\right)$$

$$=\frac{1}{2}-\frac{1}{4}+\frac{\sqrt{3}}{4\sqrt{2}}+\frac{\sqrt{3}}{4\sqrt{2}}-\frac{3}{4}$$

$$=\frac{2\sqrt{3}}{4\sqrt{2}}-\frac{1}{2}=\frac{2\sqrt{3}-2\sqrt{2}}{4\sqrt{2}}=\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}}$$

36. (a) : $\vec{u}\times\vec{v}=(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$

$\therefore\ |\vec{u}\times\vec{v}|=2|\vec{a}\times\vec{b}|$

$$=2\sqrt{[a^2b^2\sin^2\theta]}=2\sqrt{(a^2b^2-a^2b^2\cos^2\theta)}$$

$$=2\sqrt{[16-(\vec{a}\cdot\vec{b})^2]}\quad[\because |\vec{a}|=|\vec{b}|=2]$$

37. (a) : Given that $\vec{a},\vec{b},\vec{c},\vec{d}$, are vectors such that

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$ ... (i)

P_1 is the plane determined by vector $\vec{a}$ and $\vec{b}$

$\therefore$ Normal vector $\vec{n}_1=\vec{a}\times\vec{b}$

P_2 is the plane determined by vector $\vec{c}$ and $\vec{d}$

$\therefore$ Normal vector $\vec{n}_2=\vec{c}\times\vec{d}$

$\vec{n}_1\times\vec{n}_2=0\ \Rightarrow\ \vec{n}_1\,||\,\vec{n}_2$

Therefore planes are also parallel to each other.

Therefore angle between planes = 0

38. (a, b) : $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b}=\lambda$

$(b+c)\log a=\lambda(b^2-c^2)\Rightarrow\log a^{b+c}=\lambda(b^2-c^2)$

Similarly, $\log b^{c+a}=\lambda(c^2-a^2)$

$\log c^{a+b}=\lambda(a^2-b^2)$

$a^{b+c}+b^{c+a}+c^{a+b}\geq 3$

39. (b, c, d) : $x=\frac{1-\sin\theta}{\cos\theta},\ y=\frac{1+\cos\theta}{\sin\theta}$

$$xy=\frac{1-\sin\theta+\cos\theta-\cos\theta\sin\theta}{\sin\theta\cos\theta}$$

$x - y = -(xy + 1) \Rightarrow xy + x - y + 1 = 0$

$$x=\frac{y-1}{y+1},\ y=\frac{1+x}{1-x}$$

40. (b, d) : $\sin\left(\theta+\frac{\pi}{3}\right)=\frac{6x-x^2-11}{2}$

$$-1\le\sin\left(\theta+\frac{\pi}{3}\right)\le 1$$

$-2 \le 6x - x^2 - 11 \le 2$

At $x = 3$

$$\sin\left(\theta+\frac{\pi}{3}\right)=-1\Rightarrow\theta=2n\pi-\frac{5\pi}{6}$$

$$n=1,2,\ \theta=\frac{7\pi}{6},\frac{19\pi}{6}$$

$$(x,\theta)=\left(3,\frac{7\pi}{6}\right),\left(3,\frac{19\pi}{6}\right)$$

41. (a, d) : If $x\in\left(\frac{5\pi}{2},3\pi\right), \sin^{-1}(\sin x)=3\pi-x,$

$$\cos^{-1}(\cos x)=x-2\pi, \tan^{-1}(\tan x)=x-3\pi$$

42. (a, b) : $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$

Also, $(a^2 - b^2 + c^2)^2 = a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 - c^2a^2)$

$\Rightarrow (a^2 - b^2 + c^2)^2 = 2c^2a^2$

$$\Rightarrow \frac{a^2-b^2+c^2}{2ca}=\pm\frac{1}{\sqrt{2}}=\cos B$$

$\Rightarrow B = 45°$ or $135°$

43. (a, b, d) : $r=\frac{\Delta}{s}, r_1=\frac{\Delta}{s-a}$

$$r_1r=\frac{\Delta^2}{s(s-a)}=\frac{s(s-a)(s-b)(s-c)}{s(s-a)}$$

$= (s - b)(s - c) = (s - b)^2$

$$=\frac{(2s-2b)^2}{4}=\frac{(a+b+c-2b)^2}{4}$$

$$=\frac{a^2}{4}=\frac{4R^2\sin^2 A}{4}=R^2\sin^2 A$$

Also if $\angle B = \theta \Rightarrow \angle A = \pi - 2\theta$

$r_1r = R^2\sin^2(\pi - 2\theta) = R^2\sin^2 2\theta = R^2\sin^2 2B$

44. (a, b) : We have, $2s = a + b + c,$

$A^2 = s(s - a)(s - b)(s - c)$

Now, A.M. $\ge$ G.M.

$$\Rightarrow \frac{s+(s-a)+(s-b)+(s-c)}{4}>\left[s(s-a)(s-b)(s-c)\right]^{1/4}$$

$$\Rightarrow \frac{4s-2s}{4}>[A^2]^{1/4}\Rightarrow\frac{s}{2}>A^{1/2}\Rightarrow A<\frac{s^2}{4}$$

Also, $\frac{(s-a)+(s-b)+(s-c)}{3}\ge[(s-a)(s-b)(s-c)]^{1/3}$

or, $\frac{s}{3}\ge\left[\frac{A^2}{s}\right]^{1/3}$ or, $\frac{A^2}{s}\le\frac{s^3}{27}\Rightarrow A\le\frac{s^2}{3\sqrt{3}}$

45. (a, d) : $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^2$

$\Rightarrow \sin^{-1}x + \sin^{-1}w = \sin^{-1}y + \sin^{-1}z = \pi$

or, $\sin^{-1}x + \sin^{-1}w = \sin^{-1}y + \sin^{-1}z = -\pi$

$\Rightarrow x = y = z = w = 1$ or, $x = y = z = w = -1$

Hence, maximum value of $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4}\end{vmatrix}=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=2$

and minimum value $=\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}=-2$

46. (a, b, c) : (a) $\cos(\tan^{-1}(\tan(4-\pi)))=\cos(4-\pi)$

$= \cos(\pi - 4) = -\cos 4 > 0$

(b) $\sin(\cot^{-1}(\cot(4 - \pi))) = \sin(4 - \pi)$

$= -\sin 4 > 0$ (as $\sin 4 < 0$)

(c) $\tan(\cos^{-1}(\cos(2\pi - 5))) = \tan(2\pi - 5)$

$= -\tan 5 > 0$ (as $\tan 5 < 0$)

(d) $\cot(\sin^{-1}(\sin(\pi - 4))) = \cot(\pi - 4) = -\cot 4 < 0$

47. (a, b, c) : $z_1 = a + ib,\ z_1 = \cos\theta + i\sin\theta$

$z_2 = c + id,\ z_2 = \cos\alpha + i\sin\alpha$

$z_1z_2 = \cos(\theta + \alpha) + i\sin(\theta + \alpha)$

$\therefore$ $\text{Re}(z_1z_2) = \cos(\theta + \alpha) = 0$

$$\Rightarrow \theta+\alpha=\frac{\pi}{2} \qquad \ldots(i)$$

$$\omega_1=\cos\theta+i\cos\alpha=\cos\theta+i\cos\left(\frac{\pi}{2}-\theta\right)$$

$= \cos\theta + i\sin\theta = e^{i\theta}$

$\therefore$ $|\omega_1| = 1$

$$\omega_2=\sin\theta+i\sin\alpha=\sin\left(\frac{\pi}{2}-\alpha\right)+i\sin\alpha$$

$= \cos\alpha + i\sin\alpha = e^{i\alpha}$

$\therefore$ $|\omega_2| = 1$

$\omega_1\omega_2 = e^{i(\theta + \alpha)}$

$\therefore$ $\text{Re}(\omega_1\omega_2)=\cos(\theta+\alpha)=0 \quad \left(\because\ \theta+\alpha=\frac{\pi}{2}\right)$

48. (a, d) : $z_1 = 5 + 12i,\ |z_2| = 4$

$|z_1+iz_2| \le |z_1|+|z_2|=13+4=17$

$\therefore$ $|z_1+(1+i)z_2| \ge ||z_1|-|1+i||z_2||$

$=13-4\sqrt{2}$

$\therefore$ min $(|z_1 + (1 + i)z_2|) = 13-4\sqrt{2}$

$$\left|z_2+\frac{4}{z_2}\right|\le|z_2|+\frac{4}{|z_2|}=4+1=5$$

$$\left|z_2+\frac{4}{z_2}\right|\ge |z_2|-\frac{4}{|z_2|}=4-1=3$$

$$\therefore\ \max\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{3}\ \text{ and }\ \min\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{5}$$

49. (a, b) : $|z - i\,\mathrm{Re}(z)| = |z - \mathrm{Im}(z)|$
Let $z = x + iy$,
Then $|x + iy - ix| = |x + iy - y|$
$\Rightarrow x^2 + (y - x)^2 = (x - y)^2 + y^2$
i.e. $x^2 = y^2$ *i.e.* $y = \pm x$

50. (b, c, d)

51. (a, b, c) : $|z_1 + z_2| = |z_1 - z_2|$
$\Rightarrow (z_1+z_2)(\bar z_1+\bar z_2)=(z_1-z_2)(\bar z_1-\bar z_2)$
$\Rightarrow z_1\bar z_2+z_2\bar z_1=0$
$\Rightarrow \dfrac{z_1}{z_2}=-\left(\dfrac{\bar z_1}{\bar z_2}\right)\Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary. ...... (i)
Also from (i), $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$
$\Rightarrow \Delta AOB$ is a right angled triangle, at O. So, circumcentre $=\dfrac{z_1+z_2}{2}$.

52. (a, b, c) : The roots of $x^2 + x + 1 = 0$ are ω and ω^2.
So, $h(\omega) = 0$ and $h(\omega^2) = 0$
$\Rightarrow \omega f(1) + \omega^2 g(1) = 0$ and $\omega^2 f(1) + \omega g(1) = 0$
$\Rightarrow f(1) = g(1) = 0$
$\therefore\ h(1) = f(1) + g(1) = 0$

53. (a, b,c) : We have, $\alpha=\cos\dfrac{2\pi}{5}+i\sin\dfrac{2\pi}{5}$
and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$
$\therefore\ |1+\alpha+\alpha^2+\alpha^3|=|-\alpha^4|=|\alpha|^4=1.$
Also $|1+\alpha+\alpha^2| = |-\alpha^3(1+\alpha)| = |1+\alpha|$ ...(i)

$$=\left|1+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right|=\left|2\cos\frac{\pi}{5}\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right|$$

$$=2\cos\frac{\pi}{5}$$

Again from (i), $|1+\alpha|=|1+\alpha+\alpha^2|=2\cos\dfrac{\pi}{5}$

54. (b, c) : Let $a = a_1 + ia_2$ and $c = c_1 + ic_2$, then
Slope of the line $a\bar z+\bar a z+b=0$ is $-\dfrac{a_1}{a_2}$
and slope of the line $c\bar z+\bar c z+d=0$ is $-\dfrac{c_1}{c_2}$
So, $-\dfrac{a_1}{a_2}\times\dfrac{-c_1}{c_2}=-1\Rightarrow a_1c_1+a_2c_2=0$

$$\Rightarrow\left(\frac{a+\bar a}{2}\right)\left(\frac{c+\bar c}{2}\right)+\left(\frac{a-\bar a}{2i}\right)\left(\frac{c-\bar c}{2i}\right)=0\Rightarrow a\bar c+\bar a c=0$$

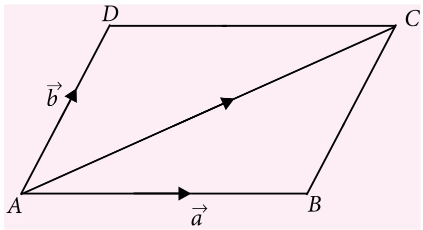
$\therefore\ a\bar c$ is purely imaginary. Also $\dfrac{a}{c}=-\dfrac{\bar a}{\bar c}\Rightarrow\dfrac{a}{c}$ is also purely imaginary.
$\Rightarrow \arg\left(\dfrac{a}{c}\right)=\pm\dfrac{\pi}{2}$

55. (a, c) : Given, $|\vec a|=1$, $|\vec c|=1$ and $|\vec b|=4$ and $\vec a\times\vec b=2\vec a\times\vec c$
Since, angle between $\vec a$ and $\vec c$ is $\cos^{-1}\left(\dfrac{1}{4}\right)$

$$\vec a\cdot\vec c=|\vec a||\vec c|\frac{1}{4}=1\cdot1\cdot\frac{1}{4}\ \Rightarrow\ \vec a\cdot\vec c=\frac{1}{4}$$

and $\vec b-2\vec c=\lambda\vec a\ \Rightarrow\ \vec b=2\vec c+\lambda\vec a$
Squaring, $b^2=4c^2+\lambda^2a^2=4\lambda\vec c\cdot\vec a\ \Rightarrow\ 16=4+\lambda^2+\lambda$
$\Rightarrow \lambda^2 + \lambda - 12 = 0 \ \Rightarrow\ \lambda = -4, 3$

56. (b, c) : $\overrightarrow{AC}=\vec a+\vec b$
Then, $|\overrightarrow{AC}| = |\vec a+\vec b|$



$$|\overrightarrow{AC}|^2=(\vec a)^2+(\vec b)^2+2\vec a\cdot\vec b$$
$$=\{(3\vec\alpha-\vec\beta)^2+(\vec\alpha+3\vec\beta)^2+2(3\vec\alpha-\vec\beta)\cdot(\vec\alpha+3\vec\beta)\}$$
$$=9\alpha^2+\beta^2-6\vec\alpha\cdot\vec\beta+\alpha^2+9\beta^2+6\vec\alpha\cdot\vec\beta+6\alpha^2-6\beta^2+16\vec\alpha\cdot\vec\beta$$
$$=16\alpha^2+4\beta^2+16\vec\alpha\cdot\vec\beta$$
$$=16\alpha^2+4\beta^2+16|\vec\alpha||\vec\beta|\cos\frac{\pi}{3}$$
$= 64 + 16 + 32 = 112$
$AC=4\sqrt7$
Similarly, $BD=4\sqrt3$

57. (a) **58. (b)** **59. (b)**
Let $\tan\theta = x$

$$\sqrt{x^2+1}+\sqrt{\frac{1}{x^2}+1}=c\Rightarrow x^2+x+1=\pm x\sqrt{c^2+1}\quad\text{... (1)}$$

If $c=\sqrt8\Rightarrow x^2+x+1=\pm3x$
$\Rightarrow x^2 - 2x + 1 = 0$ or $x^2 + 4x + 1 = 0$
$\Rightarrow x=1$ or $-2\pm\sqrt3$
$\tan\theta = 1$ or $\tan\theta=-2\pm\sqrt3<0$ (Impossible)
$\theta=\dfrac{\pi}{4},\dfrac{5\pi}{4}$, But $\theta=\dfrac{5\pi}{4}$ (not possible)

From (1), $x^2+(1+\sqrt{c^2+1})x+1=0$

or $x^2+(1-\sqrt{c^2+1})x+1=0$

The equation has real solutions if $(1+\sqrt{c^2+1})^2-4\ge 0$

and $(1-\sqrt{c^2+1})^2-4\ge 0$

$\Rightarrow (3-\sqrt{c^2+1})\le 0$ and

$(\sqrt{c^2+1}+3)(\sqrt{c^2+1}-1)\ge 0$

The 2nd inequality holds for all c^2 while the 1st inequality hold if $c^2 \geq 8$

If $c^2 < 8$ then, the equation has 2 solutions in $(0, 2\pi)$

If $c^2 > 8$ then, the equation has 4 solutions in $(0, 2\pi)$

60. (b): $\begin{cases} 2x>0 \\ (2x)^2=7x-2-2x^2 \end{cases} \Rightarrow \begin{cases} x>0 \\ 6x^2-7x+2=0 \end{cases}$

$\Rightarrow \begin{cases} x>0 \\ x=\frac{1}{2},\frac{2}{3} \end{cases}$

61. (c) : $\begin{cases} x^2-1>0 \\ 2x^2+5x>0, x\ne 1 \Rightarrow x<\frac{-5}{2} \text{ and } x>1 \\ x^3+6=2x^2+5x \end{cases}$

62. (d) : $x - 9 > 0$, $2x - 1 > 0$, $(x - 9)(2x - 1) = 10^2 = 100$

$\Rightarrow x>9, x>\frac{1}{2}, 2x^2-19x-91=0$

$\Rightarrow x > 9$ and $x = 13$, $x=-\frac{7}{2}$

$x = 13$

63. (b) **64. (c)** **65. (b)**

$8x^3 - 4x^2 - 4x + 1 = 0$ ... (1)

$x\to\frac{1}{x}\Rightarrow x^3-4x^2-4x+8=0$ ... (2)

$\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}=4$

$\Rightarrow x(x^2-4)=4(x^2-2)$

$\Rightarrow \sqrt{x+1}(x+1-4)=4(x+1-2)$,

Replacing $x^2 \to x + 1$ in (2)

$\Rightarrow x^3 - 21x^2 + 35x - 7 = 0$ ... (3)

$x\to\frac{1}{x}$ in (3) $\Rightarrow 7x^3 - 35x^2 + 21x - 1 = 0$

$\cot^2\frac{\pi}{7}+\cot^2\frac{3\pi}{7}+\cot^2\frac{5\pi}{7}=\frac{35}{7}=5$

$\left(\sum_{r=1}^{3}\tan^2\left(\frac{2r-1}{7}\right)\right)\left(\sum_{r=1}^{3}\cot^2\left(\frac{2r-1}{7}\right)\right)=21\times 5=105$

66. (c) : $\sin^6x + \cos^6x = 1 - 3\sin^2x\cos^2x$

$=1-\frac{3}{4}\sin^2 2x=1-\frac{3}{4}\left(\frac{1-\cos 4x}{2}\right)$

$=\frac{3}{8}\cos 4x+\frac{5}{8}<\frac{7}{16}\Rightarrow \frac{3}{8}\cos 4x+\frac{3}{16}<0$

$\Rightarrow \cos 4x<-\frac{1}{2}\Rightarrow 2n\pi+\frac{2\pi}{3}<4x<2n\pi+\frac{4\pi}{3}$

$\Rightarrow \frac{n\pi}{2}+\frac{\pi}{6}<x<\frac{n\pi}{2}+\frac{\pi}{3}$

67. (d) : $\cos 2x + 5\cos x + 3 \geq 0$

$\Rightarrow 2\cos^2x - 1 + 5\cos x + 3 \geq 0$

$\Rightarrow (2\cos x+1)(2+\cos x)>0\Rightarrow \cos x\ge -\frac{1}{2}$

$\Rightarrow -\frac{2\pi}{3}\le x\le \frac{2\pi}{3}$

68. (d) : $\cos\left(2x-\frac{\pi}{6}\right)\ge -\frac{1}{2}$

$2n\pi-\frac{2\pi}{3}\le 2x-\frac{\pi}{6}\le 2n\pi+\frac{2\pi}{3}$

$2n\pi-\frac{\pi}{2}\le 2x\le 2n\pi+\frac{5\pi}{6},\ n\pi-\frac{\pi}{4}\le x\le n\pi+\frac{5\pi}{12}$

69. (d) **70. (d)** **71. (b)**

72. $A \to r,s$; $B \to q$; $C \to p$; $D \to p$

73. $A \to p$; $B \to q$; $C \to r$; $D \to s$

$$f(x)=\begin{cases} \frac{7\pi}{4}-\cos^{-1}x, & -1\le x\le \frac{-1}{\sqrt{2}} \\ \frac{\pi}{4}+\cos^{-1}x, & \frac{-1}{\sqrt{2}}\le x\le 1 \end{cases}$$

74. $A \to q$; $B \to s$; $C \to r$; $D \to p$

(A) $z^2 = -|z|$

$\therefore$ $|z^2| = |z|$ or $|z|^2 - |z| = 0$

$\therefore$ $|z|(|z| - 1) = 0$ $\therefore$ $|z| = 0$ $\Rightarrow$ $z = 0$

$|z| = 1$ $\Rightarrow$ $z^2 + 1 = 0$ by (A)

$\Rightarrow$ $z^2 = -1$ $\therefore$ $z = 0 \pm i$

Hence, (A) has 3 solutions $0, \pm i$.

(B) $z = x + iy$

$\therefore$ $\bar{z}=x-iy$

$\therefore$ $z^2+\bar{z}^2=0 \Rightarrow 2(x^2-y^2)=0$

$\therefore$ $y = \pm x$

$\therefore$ $z = x(1 \pm i)$ where $x \in R$

Hence there will be infinite solutions.

(C) $z^2+8\bar{z}=0 \Rightarrow z^2=-8\bar{z}$

$\therefore$ $|z^2|=|-8\bar{z}|$ or $|z|^2=8|z|$

$\therefore$ $|z| = 0$ or $|z| = 8$

$\therefore$ $z = 0$ is one solution.

Now $|z| = 8$ $\Rightarrow$ $|z|^2=64 \Rightarrow z\bar{z}=64$

Using $z^2=-8\bar{z}=-8\cdot\frac{64}{z}$

$\therefore$ $z^3 = -(8)^3$

$\therefore\quad z = -8(1, \omega, \omega^2) = -(8, 8\omega, 8\omega^2)$

Thus, there are 1 + 3 = 4 solutions.

(D) $(x-2)^2 + (y-0)^2 = 1$

and $(x-1)^2 + y^2 = 4$

$C_1(2, 0), r_1 = 1$; $C_2(1, 0), r_2 = 2$

$\therefore\quad C_1C_2 = 1 = r_2 - r_1$

Hence the two circles touch internally at the point (3, 0). Thus there is only one solution.

$\therefore$ (D) $\to$ (p)

75. (2) : We have $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0$

$$\Rightarrow \left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)^2=0$$

$$\Rightarrow \frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}$$
$$+2\left[\frac{ab}{(b-c)(c-a)}+\frac{bc}{(c-a)(a-b)}+\frac{ca}{(a-b)(b-c)}\right]=0$$

$$\Rightarrow \frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}$$
$$+2\left[\frac{ab(a-b)+bc(b-c)+ca(c-a)}{(b-c)(c-a)(a-b)}\right]=0 \quad \text{... (i)}$$

Now $ab(a-b) + bc(b-c) + ca(c-a)$

$= a^2b - ab^2 + b^2c - bc^2 + ac^2 - a^2c$

$= a^2(b-c) - a(b^2 - c^2) + bc(b-c)$

$= (b-c)[a^2 - a(b+c) + bc]$

$= (b-c)(a-b)(a-c) = -(b-c)(c-a)(a-b)$

$$\therefore\quad \text{(i)} \Rightarrow \frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}+2(-1)=0$$

$$\Rightarrow \frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}=2$$

76. (2) : $z_1\bar{z}_1=1,\ z_2\bar{z}_2=4,\ z_3\bar{z}_3=9$

$\therefore\ |9z_1z_2+4z_1z_3+z_2z_3|$

$=|z_1z_2z_3\bar{z}_3+z_1z_3z_2\bar{z}_2+z_2z_3z_1\bar{z}_1|=12$

$\Rightarrow\ |z_1z_2z_3||\bar{z}_1+\bar{z}_2+\bar{z}_3|=12$

$\Rightarrow\ 6|z_1+z_2+z_3|=12$

77. (4) : The first relation can be written as

$$\text{Arg}\frac{3}{2}+\text{Arg}\frac{z-2-i}{z-4-3i}=\frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\frac{z-2-i}{z-4-3i}=\frac{\pi}{4} \qquad \left(\because\ \text{Arg}\frac{3}{2}=0\right)$$

$$\Rightarrow \text{Arg}\frac{x+iy-2-i}{x+iy-4-3i}=\frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\frac{(x-2)+i(y-1)}{(x-4)+i(y-3)}=\frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\frac{[(x-2)+i(y-1)][(x-4)-i(y-3)]}{(x-4)^2+(y-3)^2}=\frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\frac{\begin{bmatrix}(x-2)(x-4)+(y-1)(y-3)\\ +i\{(y-1)(x-4)-(x-2)(y-3)\}\end{bmatrix}}{(x-4)^2+(y-3)^2}$$

$$=\tan^{-1}\frac{(y-1)(x-4)-(x-2)(y-3)}{(x-2)(x-4)+(y-1)(y-3)}=\frac{\pi}{4}$$

The other equation is also a circle given by

$x^2 + y^2 - 6x + 2y + 1 = 0$

The two circles intersect at

$$\left(4-\frac{4}{\sqrt{5}}, 1+\frac{2}{\sqrt{5}}\right) \text{ and } \left(4+\frac{4}{\sqrt{5}}, 1-\frac{2}{\sqrt{5}}\right)$$

$\Rightarrow\ k = 4$

78. (7) : $f(x)=A_0+\sum_{k=1}^{20}A_kx^k$

$\Rightarrow\ f(x) + f(\alpha x) + f(\alpha^2x) + ... + f(\alpha^6x)$

$$=7A_0+\sum_{k=1}^{20}A_kx^k(1+\alpha^k+\alpha^{2k}+\alpha^{3k}+....+\alpha^{6k})$$

In the summation when $k \neq 7$, $k \neq 14$ then

$$1+\alpha^k+\alpha^{2k}+...+\alpha^{6k}=\frac{\alpha^{7k}-1}{\alpha^k-1}=0 \quad (\because\ \alpha^{7k}=1, \alpha^k\neq 1)$$

when $k = 7$, $k = 14$, the terms in summation $= 7A_7x^7$ and $7A_{14}\,x^{14}$ respectively.

Thus $f(x) + f(\alpha x) + ... + f(\alpha^6x)$

$= 7(A_0 + A_7x^7 + A_{14}x^{14})$

$\Rightarrow\ k = 7$

79. (9) : The resulting complex number must be

$3(4-3i)(\cos\pi + i\sin\pi) = 3(4-3i)(-1)$

$= -12 + 9i$

$\Rightarrow\ \lambda = 9$

80. (3) : $|z|=\left|z-\frac{2}{z}+\frac{2}{z}\right|\le\left|z-\frac{2}{z}\right|+\frac{2}{|z|}=2+\frac{2}{|z|}$

$\Rightarrow\ |z|^2 - 2|z| - 2 \le 0$

Solving the quadratic, we get

$|z|\le 1+\sqrt{3}\ \Rightarrow\ \lambda=3$

81. (3) : z_1, z_2, z_3 form equilateral triangle if and only if $z_1^2+z_2^2+z_3^2=\Sigma z_1z_2$

Take $z_3 = 0$

$\therefore\quad z_1^2+z_2^2+z_3^2=z_1z_2$

$\Rightarrow\ (z_1 + z_2)^2 = 3z_1z_2$

$\Rightarrow\ a^2 = 3b.$

Problems from AROUND THE WORLD

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1. Prove that for any integer $n > 1$ the sum S of all divisors of n (including 1 and n) satisfies the inequalities $k\sqrt{n} < S < \sqrt{2kn}$, where k is the number of divisors of n.

2. A circle is inscribed in the trapezoid $ABCD$. Let K, L, M, N be the points of intersections of the circle with diagonals AC and BD respectively (K is between A and L & M is between B and N). Given that $AK \cdot LC = 16$ and $BM \cdot ND = \frac{9}{4}$, find the radius of the circle.

3. A convex quadrilateral $ABCD$ is inscribed in a circle whose center O is inside the quadrilateral. Let $MNPQ$ be the quadrilateral whose vertices are the projections of the intersection point of the diagonals AC and BD onto the sides of $ABCD$. Prove that $2[MNPQ] \le [ABCD]$.

4. A rectangular parallelopiped has integer dimension. All of its faces are painted green. The parallelopiped is partitioned into unit cubes by planes parallel to its faces. Find all possible measurements of the parallelopiped if the number of cubes without a green face is one third of the total number of cubes.

5. Let $\{a_n\}$ be sequence of integers such that for $n \ge 1$. $(n-1)a_{n+1} = (n+1)a_n - 2(n-1)$. If 2000 divides a_{1999}, find the smallest $n \ge 2$ such that 2000 divides a_n.

6. Let x, y, z be non negative real numbers such that $x + y + z = 1$. Prove that $x^2y + y^2z + z^2x \le \frac{4}{27}$, and determine when equality occurs.

7. Let a be a real number. Let $\{f_n(x)\}$ be a sequence of polynomials such that $f_0(x) = 1$ and
$f_{n+1}(x) = x f_n(x) + f_n(ax)$ for $n = 0,1,2,\ldots$

(a) Prove that $f_n(x) = x^n f_n\left(\frac{1}{x}\right)$
for $n = 0,1,2,\ldots$

(b) Find an explicit expression for $f_n(x)$.

8. Show that there exists a triangle ABC for which, with the usual labelling of sides and medians, it is true that $a \ne b$ and $a + m_a = b + m_b$. Show further that there exists a number k such that for each such triangle $a + m_a = b + m_b = k(a + b)$. Finally, find all possible ratios $a : b$ of the sides of these triangles.

9. If $f : R \to R$ is a function satisfying for all $x \in R$, $f(-x) = -f(x)$,

$$f(x+1) = f(x) + 1 \text{ and } f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \text{ (when } x \ne 0).$$

prove that $f(x) = x$ for all real values of x.

10. M is an interior point of a triangle ABC. Bisectors of interior angles BMC, CMA, AMB intersect BC, CA, AB respectively at X, Y, Z. Show that AX, BY, CZ are concurrent; if P is the point of concurrence and $\frac{PA}{PX} \cdot \frac{PB}{PY} \cdot \frac{PC}{PZ} = 8$, also show that M is the circumcenter of ΔABC.

11. Show that in any triangle ABC

$\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}$ with equality holding if and only if the triangle is equilateral.

12. For a positive integer n, define $A(n)$ to be

$$\frac{(2n)!}{(n!)^2}$$

Determine the sets of positive integers n for which

i) $A(n)$ is an even number

ii) $A(n)$ is a multiple of 4.

13. Find all cubic polynomials $p(x)$ such that $(x-1)^2$ is a factor of $p(x) + 2$ and $(x+1)^2$ is a factor of $p(x) - 2$.

14. There are ten objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceed 10, prove that the ten objects can be divided into two groups that balance each other when placed on the two pans of a balance.

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SOLUTIONS

1. Let the divisors of n be $1 = d_1 < d_2 <..... < d_k = n$, so that $d_i d_{k+1-i} = n$ for each i. Then

$$S=\sum_{i=1}^{k} d_i=\sum_{i=1}^{k}\frac{d_i+d_{k+1-i}}{2}>\sum_{i=1}^{k}\sqrt{d_i d_{k+1-i}}=k\sqrt{n},$$

where the inequality is strict because equality does not hold for $\frac{d_1+d_k}{2}\ge\sqrt{d_1 d_k}$.

For the right inequality, let $S_2=\sum_{i=1}^{k} d_i^2$. By the Power Mean Inequality,

$$\frac{S}{k}=\frac{\sum_{i=1}^{k} d_i}{k}\le\sqrt{\frac{\sum_{i=1}^{k} d_i^2}{k}}=\sqrt{\frac{S_2}{k}}$$

Hence, $S\le\sqrt{kS_2}$. Now

$$\frac{S_2}{n^2}=\sum_{i=1}^{k}\frac{d_i^2}{n^2}=\sum_{i=1}^{k}\frac{1}{d_{k+1-i}^2}\le\sum_{j=1}^{n}\frac{1}{j^2}<\frac{\pi^2}{6}$$

because $d_1,....., d_k$ are distinct integers between 1 and n. Therefore

$$S\le\sqrt{kS_2}<\sqrt{\frac{kn^2\pi^2}{6}}<\sqrt{2k}n$$

2. Let the circle touch sides AB, BC, CD, DA at P, Q, R, S, respectively, and let r be the radius of the circle. Let $w = AS = AP$, $x = BP = BQ$, $y = CQ = CR$, and $z = DR = DS$. We have $wz = xy = r^2$ and thus $wxyz = r^4$. $AK \cdot LC$ depends only on r and $AP \cdot CR$ and $BM \cdot ND$ depends only on r and $BP \cdot DR$.

Now, draw a parallelogram A', B', C', D' circumscribed about the same circle, with points P', Q', R', S' defined analogously to P, Q, R, S, such that $A'P'=C'R'=\sqrt{wy}$. Draw points K', L', M', N' analogously to K, L, M, N. Because $A'P' \cdot C'R' = wy$, by the observation in the first paragraph we must have $A'K' \cdot L'C' = AK \cdot LC = 16$. Therefore, $A'K' = L'C' = 4$. Also as with quadrilateral $ABCD$, we have $A'P' \cdot B'P' \cdot C'R' \cdot D'R' = r^4 = wxyz$. Thus, $B'P' \cdot D'R' = xz$, and again by the observation we must have $B'M'\cdot N'D'=BM\cdot ND=\frac{9}{4}$. Therefore, $B'M'=N'D'=\frac{3}{2}$.

Letting O be the center of the circle, we have $A'O = 4 + r$ and $S'O = r$. By the Pythagorean Theorem, $A'S'=\sqrt{8r+16}$. Similarly, $S'D'=\sqrt{3r+\frac{9}{4}}$. Because $A'S' \cdot S'D' = r^2$, we have $(8r+16)\left(3r+\frac{9}{4}\right)=r^4$, which has positive solution $r = 6$ and, by Descartes' rule of signs, no other positive solutions.

3. The result actually holds even when $ABCD$ is not cyclic. We begin by proving the following result:

Lemma. If $\overline{XW}$ is an altitude of triangle XYZ, then

$$\frac{XW}{YZ}\le\frac{1}{2}\tan\left(\frac{\angle Y+\angle Z}{2}\right)$$

Proof: X lies on an arc of a circle determined by $\angle YXZ = 180° - \angle Y - \angle Z$. Its distance from $\overline{YZ}$ is maximized when it is at the center of this arc, which occurs when $\angle Y = \angle Z$.

At this point, $\frac{XW}{YZ}=\frac{1}{2}\tan\left(\frac{\angle Y+\angle Z}{2}\right)$.

Suppose M, N, P, Q are on sides AB, BC, CD, DA, respectively. Also let T be the intersection of $\overline{AC}$ and $\overline{BD}$.

Let $\alpha = \angle ADB$, $\beta = \angle BAC$, $\gamma = \angle CAD$, $\delta = \angle DBA$. From the lemma,

$$MT\le\frac{1}{2}AB\cdot\tan\left(\frac{\beta+\delta}{2}\right)\text{ and }QT\le\frac{1}{2}AD\cdot\tan\left(\frac{\alpha+\gamma}{2}\right)$$

Also, $\angle MTQ = 180° - \angle QAM = 180° - \angle DAB$.

$$\text{Thus } 2[MTQ]=MT\cdot QT\sin\angle MTQ$$
$$\le\frac{1}{4}\tan\left(\frac{\alpha+\gamma}{2}\right)\tan\left(\frac{\beta+\delta}{2}\right)AB\cdot AD\sin\angle DAB$$

Because $\frac{\alpha+\gamma}{2}+\frac{\beta+\delta}{2}=90°$, this last expression exactly equals

$$\frac{1}{4}AB.\ AD\sin\angle DAB=\frac{1}{2}[ABD]$$

Thus, $2[MTQ]\le\frac{1}{2}[ABD]$

Likewise, $2[NTM]\le\frac{1}{2}[BCA]$, $[PTN]\le\frac{1}{2}[CDB]$, and $[QTP]\le\frac{1}{2}[DAC]$. Adding these four inequalities shows that $2[MNPQ]$ is at most

$\frac{1}{2}([ABD]+[CDB])+\frac{1}{2}([BCA]+[DAC])=[ABCD]$, as desired.

4. Let the parallelopiped's dimensions be a, b, c. These lengths must all be at least 3 or else every cube has a green face. The given condition is equivalent to $3(a - 2)(b - 2)(c - 2) = abc$,

$$\text{or }\ 3=\frac{a}{a-2}\cdot\frac{b}{b-2}\cdot\frac{c}{c-2}$$

If all the dimensions are at least 7, then

$$\frac{a}{a-2}\cdot\frac{b}{b-2}\cdot\frac{c}{c-2}\le\left(\frac{7}{5}\right)^3=\frac{343}{125}<3,$$

a contradiction. Thus one of the dimensions – say, a – equals 3, 4, 5, or 6. Assume without loss of generality that $b \le c$.

When $a = 3$ we have $bc = (b - 2)(c - 2)$, which is impossible.

When $a = 4$, rearranging the equation yields $(b - 6)(c - 6) = 24$. Thus $(b, c) = (7,30), (8, 18), (9, 14)$, or $(10, 12)$.

When $a = 5$, rearranging the equation yields $(2b - 9)(2c - 9) = 45$.

Thus $(b, c) = (5, 27), (6, 12)$, or $(7, 9)$.

Finally, when $a = 6$, rearranging the equation yields $(b - 4)(c - 4) = 8$.

Thus $(b, c) = (5, 12)$ or $(6, 8)$.

Therefore, the parallelopiped may measure $4 \times 7 \times 30$, $4 \times 8 \times 18$, 9×14, $4 \times 10 \times 12$, 5×527, $5 \times 6 \times 12$, $5 \times 7 \times 9$ or $6 \times 6 \times 8$.

5. First, we note that the sequence $a_n = 2n - 2$ works. Then writing $b_n = a_n - (2n - 2)$ gives the recursion $(n - 1)b_{n+1} = (n + 1)b_n$.

For $n \ge 2$, observe that

$$b_n=b_2\cdot\prod_{k=2}^{n-1}\frac{k+1}{k-1}=b_2\cdot\frac{\prod_{k=3}^{n}k}{\prod_{k=1}^{n-2}k}=\frac{n(n-1)}{2}b_2$$

Thus when $n \ge 2$, the solution to the original equation is of the form $a_n=2(n-1)+\frac{n(n-1)}{2}c$ for some constant c.

Plugging in $n = 2$ shows that $c = a_2 - 2$ is an integer. Now, because

$2000\,|\,a_{1999}$ we have $2(1999-1)+\frac{1999\cdot 1998}{2}.c\equiv 0$

$\Rightarrow -4 + 1001c \equiv 0 \Rightarrow c \equiv 4 \pmod{2000}$.

Then $2000|a_n$ exactly when

$2(n - 1) + 2n(n - 1) \equiv 0 \pmod{2000}$

$\Leftrightarrow (n - 1)(n + 1) \equiv 0 \pmod{1000}$

$(n - 1)(n + 1)$ is divisible by 8 exactly when n is odd, and it is divisible by 125 exactly when either $n - 1$ or $n + 1$ is divisible by 125. The smallest $n \ge 2$ satisfying these requirements is $n = 249$.

6. Assume without loss of generality that

$x = \max\{x, y, z\}$

If $x \ge y \ge z$, then

$x^2y + y^2z + z^2x \le x^2y + y^2z + z^2x + z(xy + (x - y)(y - z))$

$$=(x+z)^2y=4\left(\frac{1}{2}-\frac{1}{2}y\right)\left(\frac{1}{2}-\frac{1}{2}y\right)y\le\frac{4}{27},$$

where the last inequality follows from the A.M.-G.M. inequality. Equality occurs if and only if $z = 0$ (from the first inequality) and $y=\frac{1}{3}$, in which case $(x, y, z)=\left(\frac{2}{3},\frac{1}{3},0\right)$. If, $x \ge z \ge y$, then

$x^2y + y^2z + z^2x = x^2z + z^2y + y^2x - (x - z)(z - y)(x - y)$

$$\le x^2z+z^2y+y^2x\le\frac{4}{27},$$

where the second inequality is true from the result we proved for $x \ge y \ge z$ (except with y and z reversed). Equality holds in the first inequality only when two of x, y, z are equal, and in the second inequality only when $(x,z,y)=\left(\frac{2}{3},\frac{1}{3},0\right)$. Because these conditions can't both be true, the inequality is actually strict in this case.

Therefore the inequality is indeed true, and equality holds when (x, y, z) equals

$\left(\frac{2}{3},\frac{1}{3},0\right),\left(\frac{1}{3},0,\frac{2}{3}\right)$, or $\left(0,\frac{2}{3},\frac{1}{3}\right)$

7. When $a = 1$, we have $f_n(x) = (x + 1)^n$ for all n, and part (a) is easily checked. Now assume that $a \ne 1$.

Observe that f_n has degree n and always has constant term 1. Write $f_n(x) = c_0 + c_1x + ... + c_nx^n$. We prove by induction on n that $(a^i - 1)c_i = (a^{n+1-i} - 1)c_{i-1}$ for $0 \le i \le n$ (where we let $c_{-1} = 0$).

The base case $n = 0$ is clear. Now suppose that

$$f_{n-1}(x)=b_0+b_1x+.....+b_{n-1}x^{n-1}$$

satisfies the claim: specifically, we know

$(a^i - 1)b_i = (a^{n-i} - 1)b_{i-1}$ and

$(a^{n+1-i}-1)b_{i-2}=(a^{i-1}-1)b_{i-1}$ for $i\ge 1$

For $i = 0$, the claim states $0 = 0$. For $i \ge 1$, the given recursion gives $c_i=b_{i+1}a^ib_i$ and $c_{i-1}=b_{i-2}+a^{i-1}b_{i-1}$.

Then the claim is equivalent to

$$(a^i-1)c_i=(a^{n+1-i}-1)c_{i-1}\Leftrightarrow(a^i-1)(b_{i-1}+a^ib_i)$$

$$=(a^{n+1-i}-1)(b_{i-2}+a^{i-1}b_{i-1})$$

$$\Leftrightarrow\ (a^i-1)b_{i-1}+a^i(a^i-1)b_i$$

$$=(a^{n+1-i}-1)b_{i-2}+(a^n-a^{i-1})b_{i-1}$$

$$\Leftrightarrow\ (a^i-1)b_{i-1}+a^i(a^{n-i}-1)b_{i-1}$$

$$=(a^{i-1}-1)b_{i-1}+(a^n-a^{i-1})b_{i-1}$$

$\Leftrightarrow (a^n-1)b_{i-1}=(a^n-1)b_{i-1}$,

so it is true.

Now by telescoping products, we have

$$c_i=\frac{c_i}{c_0}=\prod_{k=1}^{i}\frac{c_k}{c_{k-1}}$$

$$=\prod_{k=1}^{n}\frac{a^{n+1-k}-1}{a^k-1}=\frac{\prod_{k=n+1-i}^{n}(a^k-1)}{\prod_{k=1}^{i}(a^k-1)}$$

$$=\frac{\prod_{k=i+1}^{n}(a^k-1)}{\prod_{k=1}^{n-i}(a^k-1)}=\prod_{k=1}^{n-i}\frac{a^{n+1-k}-1}{a^k-1}$$

$=\prod_{k=1}^{n-i}\frac{c_k}{c_{k-1}}=\frac{c_{n-i}}{c_0}=c_{n-i}$, giving our explicit form.

Also, $f_n(x)=x^n f_n\left(\frac{1}{x}\right)$ if and only if $c_i = c_{n-i}$ for $i = 0, 1,\ldots,n$, and from above this is indeed the case. This completes the proof.

8. We know that

$$m_a^2=\frac{1}{4}(2b^2+2c^2-a^2),\ m_b^2=\frac{1}{4}(2a^2+2c^2-b^2),$$

$$\text{so, } m_a^2-m_b^2=\frac{3}{4}(b^2-a^2).$$

If the desired condition $m_a - m_b = b - a \neq 0$ is to be satisfied, then necessity $m_a+m_b=\frac{3}{4}(b+a)$. Find the system of equations

$m_a - m_b = b - a$

$m_a+m_b=\frac{3}{4}(b+a)$

we find that we would then have

$$m_a=\frac{1}{8}(7b-a),\ m_b=\frac{1}{8}(7a-b),$$

and $a+m_a=b+m_b=\frac{7}{8}(a+b)$

Thus $k=\frac{7}{8}$

Now we examine for what $a \neq b$ there exists a triangle ABC with sides a, b and medians $m_a=\frac{1}{8}(7b-a), m_b=\frac{1}{8}(7a-b)$. We can find all three side lengths of the triangle AB_1G, where G is the centroid of the triangle ABC and B_1 is the midpoint of the side AC

$$AB_1=\frac{b}{2},\ AG=\frac{2}{3}m_a=\frac{2}{3}\cdot\frac{1}{8}(7b-a)=\frac{1}{12}(7b-a)$$

$$B_1G=\frac{1}{3}m_b=\frac{1}{3}\cdot\frac{1}{8}(7a-b)=\frac{1}{24}(7a-b)$$

Examining the triangle inequalities for these three lengths, we get the condition $\frac{1}{3}<\frac{a}{b}<3$, from which the value $\frac{a}{b}=1$ has to be excluded by assumption. This condition is also sufficient: once the triangle AB_1G has been constructed, it can always be completed to a triangle ABC with $b = AC$, $m_a = AA_1$, $m_b = BB_1$. Then from the equality $m_a^2-m_b^2=\frac{3}{4}(b^2-a^2)$ we would also have $a = BC$.

9. Put $f(x) = x + g(x)$; then g has the properties : $g(-x) = -g(x)$,

$g(x+1)=g(x)$ and when $x\neq 0, g\left(\frac{1}{x}\right)=\frac{g(x)}{x^2}$

We have to show that $g(x) = 0$ for all $x \in R$.

$g(0) = -g(0) \Rightarrow g(0) = 0$

$g(-1) = g(-1 + 1) = g(0) = 0$

When x is neither 0 nor -1,

$$g(x)=x^2g\left(\frac{1}{x}\right)=x^2g\left(\frac{x+1}{x}\right)$$

$$=x^2\frac{(x+1)^2}{x^2}g\left(\frac{x}{x+1}\right)=(x+1)^2g\left(\frac{x}{x+1}-1\right)$$

$$=(x+1)^2g\left(\frac{-1}{x+1}\right)=-(x+1)^2g\left(\frac{1}{x+1}\right)$$

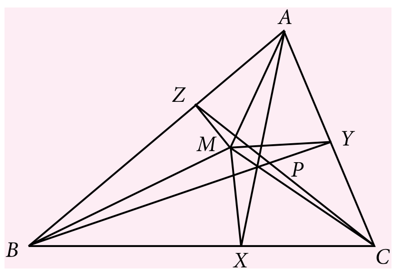
$$=-g(x+1)=-g(x)$$

Therefore $g(x) = 0$.

10. Since MX is the internal bisector of $\angle BMC, \frac{BX}{XC}=\frac{MB}{MC}$;

Similarly $\frac{CY}{YA}=\frac{MC}{MA}$ and $\frac{AZ}{ZB}=\frac{MA}{MB}$

therefore $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$



Hence by Ceva theorem, AX, BY, CZ are concurrent.
Let $\Delta BPC = t_1$, $\Delta APC = t_2$ and $\Delta APB = t_3$.

Then $\frac{PA}{PX}=\frac{\Delta APB}{\Delta BPX}=\frac{\Delta APC}{\Delta PXC}=\frac{\Delta APB+\Delta APC}{\Delta BPX+\Delta PXC}=\frac{t_2+t_3}{t_1}$

Similarly $\frac{PB}{PY}=\frac{t_3+t_1}{t_2}$ and $\frac{PC}{PZ}=\frac{t_1+t_2}{t_3}$

Now $8=\frac{(t_2+t_3)(t_3+t_1)(t_1+t_2)}{t_1t_2t_3}\geq\frac{2\sqrt{t_2t_3}\cdot 2\sqrt{t_3t_1}.2\sqrt{t_1t_2}}{t_1t_2t_3}$

(Equality holds only when $t_1 = t_2 = t_3 = 8$)

Thus $t_1=t_2=t_3$. Now $\frac{BX}{XC}=\frac{\Delta ABX}{\Delta ACX}$

$=\frac{\Delta PBX}{\Delta PCX}=\frac{\Delta ABX-\Delta PBX}{\Delta ACX-\Delta PCX}=\frac{t_3}{t_2}=1$

i.e., $BX = XC$. Since MB is the bisector of $\angle BMC$, it follows that $MB = MC$. Similarly $MC = MA$. Therefore M is the circumcenter. (Note that P is the centroid).

11. **1st Solution:** Since $\sin A=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$

$\sin B=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ca}$

and $\sin C=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab}$

what we have to prove is equivalent to the following:

$\frac{s^3(s-a)^3(s-b)^3(s-c)^3}{a^4b^4c^4}\leq\frac{3^3}{8^4}$

(Equality holds only when $a = b = c$)

Putting $x = s - a$, $y = s - b$, $z = s - c$, the above inequality becomes the following: If x, y, z are positive reals, then

$$(x+y+z)^3x^3y^3z^3\leq\frac{3^3}{8^4}(x+y)^4(y+z)^4(z+x)^4$$

(Equality holds only when $x = y = z$)

If α, β, γ are reals, we know that

$\alpha\beta + \beta\gamma + \gamma\alpha \leq \alpha^2 + \beta^2 + \gamma^2$.

(Equality holds only when $\alpha = \beta = \gamma$)

From this it follows that

$3(\alpha\beta + \beta\gamma + \gamma\alpha) \leq (\alpha + \beta + \gamma)^2$.

Replacing α, β, γ by xy, yz, zx respectively, we get

$3(x + y + z)xyz \leq (xy + yz + zx)^2$ ...(1)

Also $x+y\geq 2\sqrt{xy}, y+z\geq 2\sqrt{yz}$ and $z+x\geq 2\sqrt{zx}$

$\Rightarrow$ $8xyz \leq (x + y)(y + z)(z + x)$ ...(2)

Now $3(x + y + z)^3xyz$

$\leq (xy + yz + zx)^2 (x + y + z)^2$ (by (1))

$= ((x + y)(y + z)(z + x) + xyz)^2$

$\leq(x+y)^2(y+z)^2(z+x)^2\left(1+\frac{1}{8}\right)^2$ (by (2)),

Again by (2)

$(x+y+z)^3x^3y^3z^3\leq\frac{3^3}{8^4}(x+y)^4(y+z)^4(z+x)^2$

Equality holds only when $x = y = z$.

2nd Solution: First let us prove the following. For any three angles α, β and γ such that $\alpha + \beta + \gamma = 2k\pi$ where $k \in Z$,

$\sin\alpha+\sin\beta+\sin\gamma\leq\frac{3\sqrt{3}}{2}$ ...(1)

We can choose three unit vectors $\overline{e_1}$, $\overline{e_2}$ and $\overline{e_3}$ such that

$\angle(\overline{e_1},\overline{e_3})=\alpha, \angle(\overline{e_2},\overline{e_3})=\beta$ and $\angle(\overline{e_3},\overline{e_1})=\gamma$. Then

$(\overline{e_1}+\overline{e_2}+\overline{e_3}).(\overline{e_1}+\overline{e_2}+\overline{e_3})\geq 0$

$\Rightarrow 3+2(\overline{e_1}.\overline{e_2})+2(\overline{e_2}.\overline{e_2})+2(\overline{e_3}.\overline{e_1})\geq 0$

$\Rightarrow \cos\alpha+\cos\beta+\cos\gamma\geq-\frac{3}{2}$ ...(2)

In (2), α, β and γ can be replaced by

$\alpha+\frac{2\pi}{3},\beta+\frac{2\pi}{3}$ and $\gamma+\frac{2\pi}{3}$ yielding

$\cos\left(\alpha+\frac{2\pi}{3}\right)+\cos\left(\beta+\frac{2\pi}{3}\right)+\cos\left(\gamma+\frac{2\pi}{3}\right)\geq-\frac{3}{2}$;

i.e., $-\frac{\sqrt{3}}{2}(\sin\alpha+\sin\beta+\sin\gamma)$

$\geq-\frac{3}{2}+\frac{1}{2}(\cos\alpha+\cos\beta+\cos\gamma)\geq-\frac{9}{4}$ (by (2))

Thus we get (1). Now putting $\alpha = 2A$, $\beta = 2B$ and $\gamma = 2C$ in (1) and using the fact that

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

we get

$\sin A\sin B\sin C\leq\frac{3\sqrt{3}}{8}$ ....(3)

A little work is needed to find when we have equality in (3). When (2) becomes equality, $\overline{e_1}+\overline{e_2}+\overline{e_3}=0$ from which one can deduce that

$\overline{e_1}.\overline{e_2}=\overline{e_2}.\overline{e_3}=\overline{e_3}.\overline{e_1}$;

i.e., $\cos\alpha = \cos\beta = \cos\gamma$

Next when equality holds in (1) we should have

$\cos\left(\alpha+\frac{2\pi}{3}\right)=\cos\left(\beta+\frac{2\pi}{3}\right)=\cos\left(\gamma+\frac{2\pi}{3}\right)$

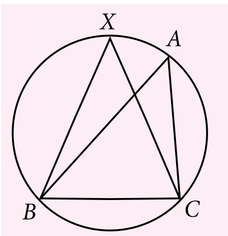
and $\cos\alpha = \cos\beta = \cos\gamma$,
Further, equality in (3) yields
$\sin 2A = \sin 2B = \sin 2C$
Since $2A + 2B + 2C = 2\pi$, we should have

$2A + 2B = \pi$ or $A = B$
$2B + 2C = \pi$ or $B = C$
$2C + 2A = \pi$ or $C = A$.

Again using $A + B + C = \pi$, we get $A = B = C$.

3rd Solution : For any triangle Δ, let $p(\Delta)$ denote the product of its sides. Let $\Im$ be the set of all triangles inscribed in a circle of diameter 1. What we have to prove is equivalent to the following: For any triangle $\Delta \in \Im, p(\Delta) \le \frac{3\sqrt{3}}{8}$, equality holding only when Δ is equilateral.

First let us show that if a triangle $\Delta \in \Im$ is not equilateral then there exists a triangle Δ' in $\Im$ such that $P(\Delta) < P(\Delta)'$.



Suppose $\Delta ABC \in \Im$ such that $AB \ne AC$. Let X be the midpoint of the arc BAC.

Then $\Delta ABC < \Delta XBC$

$\Rightarrow$ $AB.AC.\sin \angle BAC < XB{\cdot}XC \sin \angle BXC$

$\Rightarrow$ $p(ABC) < p(XBC)$.

Since $\{p(\Delta) : \Delta \in \Im\}$ is a bounded set of real numbers, we can choose a sequence of triangles $(A_nB_nC_n)_{n=1}^{\infty}$ in $\Im$ such that $p(A_nB_nC_n)$ converges to the least upper bound of $\{p(\Delta) : \Delta \in \Im\}$ and also A_n, B_n and C_n are convergent.

Let $\lim\limits_{n\to\infty} A_n = A$, $\lim\limits_{n\to\infty} B_n = B$ and $\lim\limits_{n\to\infty} C_n = C$. Then A, B, C are on the circle and $p(ABC) = \lim\limits_{n\to\infty} p(A_nB_nC_n)$.

Thus $p(ABC)$ being maximum, ABC is equilateral and

$$p(ABC) = \left(\sin\frac{\pi}{3}\right)^3 = \frac{3\sqrt{3}}{8}.$$

4th Solution: For any $\alpha, \beta, \gamma \in R$, let
$\theta(\alpha, \beta, \gamma) = \sin\alpha \sin\beta \sin\gamma$

Let us prove the following: If $\alpha, \beta, \alpha', \beta', \gamma \in [0, \pi]$ such that $\alpha \le \beta$, $\alpha', \beta', \gamma \in [\alpha, \beta]$ and $\alpha + \beta = \alpha' + \beta'$, then $\theta(\alpha, \beta, \gamma) \le \theta(\alpha', \beta', \gamma)$.

Proof : Assume $\alpha' \le \beta'$. Then

$0 \le \alpha \le \alpha' \le \beta' \le \beta \le \pi$
$\Rightarrow$ $0 \le \beta' - \alpha' \le \beta - \alpha \le \pi$
$\Rightarrow$ $\cos(\beta - \alpha) \le \cos(\beta' - \alpha')$
($\because$ $\cos x$, $x \in [0, \pi]$ is a decreasing function)
$\Rightarrow$ $\cos(\beta - \alpha) - \cos(\alpha + \beta) \le \cos(\beta' - \alpha') - \cos(\alpha' + \beta')$
$\Rightarrow$ $\sin\alpha \sin\beta \le \sin\alpha' \sin\beta'$
$\Rightarrow$ $\theta(\alpha, \beta, \gamma) \le \theta(\alpha', \beta', \gamma)$

Note that equality holds only when either $\{\alpha, \beta\} = \{\alpha', \beta'\}$ or $\gamma \in \{0, \pi\}$. Note that for any $\alpha, \beta, \gamma \in [0, \pi]$,

$$\theta(\alpha,\beta,\gamma) \le \theta\left(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}, \gamma\right)$$

Now suppose $\alpha, \beta, \gamma \in [0, \pi]$ such that $\alpha + \beta + \gamma = \pi$. We can assume $\alpha \le \frac{\pi}{3} \le \beta$; let $\alpha' = \frac{\pi}{3}$ and $\beta' = \alpha + \beta - \frac{\pi}{3}$; then $\alpha + \beta = \alpha' + \beta'$ and $\alpha', \beta' \in [\alpha, \beta]$. Therefore

$$\theta(\alpha, \beta, \gamma) \le \theta(\alpha', \beta', \gamma)$$
$$\le \theta\left(\alpha', \frac{\beta'+\gamma}{2}, \frac{\beta'+\gamma}{2}\right)$$
$$= \theta\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{8}$$

Obviously $\theta(\alpha,\beta,\gamma) = \frac{3\sqrt{3}}{8} \Leftrightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$

Remark : The fourth proof leads to the following natural generalization. If $\alpha_1, \alpha_2 \,.....,\, \alpha_n \in [0, \pi]$ such that $\alpha_1 + \alpha_2 + ... + \alpha_n = \pi$, then

$$\sin\alpha_1 \sin\alpha_2 ... \sin\alpha_n \le \sin^n\left(\frac{\pi}{n}\right)$$

Equality holds only when $\alpha_1 = \alpha_2 = ... = \alpha_n$.

12. Since the product of k consecutive integers is divisible by $k!$, $A(n)$ is an integer. We compare the highest powers of 2 dividing the numerator and denominator to determine the nature of $A(n)$.

Suppose we express n in the base 2, say,
$n = a_l2^l + a_{l-1}2^{l-1} + a_{l-2}2^{l-2} + ... + a_1.2 + a_0$, $a_l = 1$.

The highest power of 2 dividing n! is given by

$$s = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + ... + \left[\frac{n}{2^\ell}\right]$$

where $[x]$ denotes the largest integer smaller than x.

But, for $1 \le m \le l$,

$$\left[\frac{n}{2^m}\right] = a_\ell 2^{\ell-m} + a_{\ell-1}2^{\ell-m-1} + ... + a_m$$

Thus

$$s = \sum_{m=1}^{\ell}\left[\frac{n}{2m}\right]$$

$$= \sum_{m=1}^{\ell} \sum_{k=m}^{\ell} a_k 2^{k-m}$$

$$= \sum_{k=1}^{\ell} a_k \left(\sum_{m=1}^{\ell} 2^m - 1 \right)$$

$$= \sum_{k=1}^{\ell} a_k (2^k - 1)$$

$$= \sum_{k=0}^{\ell} a_k 2^k - \sum_{k=0}^{\ell} a_k$$

$= n -$ { sum of the digits of n in the base 2}

Hence, the highest power of 2 dividing $(n!)^2$ is $2s = 2n - 2$(sum of the digits of n in the base 2). Similarly the highest power of 2 dividing $(2n)!$ is $t = 2n -$ (sum of the digits of $2n$ in the base 2). But the digits of n in base 2 and those of $2n$ in base 2 are the same except for a zero at the end of the representation for $2n$.

Thus

$$t - 2s = a_l + a_{l-1} + a_{l-2} + ... + a_1 + a_0$$

where the a_i are the digits of n in base 2. Note that $a_l = 1$. Hence $t - 2s \geq 1$. But $A(n)$ is even if and only if $t - 2s \geq 1$. Hence it follows that $A(n)$ is even for all n.

Moreover $A(n)$ is divisible by 4 if and only if $t - 2s \geq 2$. Since $A(1) = 2$, 4 does not divide $A(1)$. Suppose $n = 2^l$ for some l. Then $a_l = 1$ and $a_i = 0$ for $0 \leq i \leq l - 1$. Hence $t - 2s = 1$ and $A(n)$ is not divisible by 4. On the other hand if n is not a power of 2, then for some l,

$$n = 2^l + a_{l-1}2^{l-1} + a_{l-2}2^{l-2} + ... + a_0$$

where $a_i \neq 0$ for at least one i and hence must be equal to 1. Thus $t - 2s \geq 1 + a_i \geq 2$. It follows that $A(n)$ is divisible by 4 if and only if n is not a power of 2.

Remark : Given any prime p, the highest power s of p dividing $n!$ is given by

$$s = \frac{n - (\text{sum of the digits of } n \text{ in the base } p)}{p-1}$$

13. If $(x - \alpha)$ divides a polynomial $q(x)$ then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since $(x - 1)$ divides $p(x) + 2$, we get

$$a + b + c + d + 2 = 0.$$

Hence $d = -a - b - c - 2$ and

$$p(x) + 2 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$$
$$= (x - 1)\{a(x^2 + x + 1) + b(x + 1) + c\}$$

Since $(x - 1)^2$ divides $p(x) + 2$, we conclude that $(x - 1)$ divides $a(x^2 + x + 1) + b(x + 1) + c$. This implies that $3a + 2b + c = 0$. Similarly, using the information that $(x + 1)^2$ divides $p(x) - 2$, we get two more relations : $-a + b - c + d - 2 = 0$; $3a - 2b + c = 0$. Solving these for a, b, c, d, we obtain $b = d = 0$, and $a = 1, c = -3$. This is only one polynomial satisfying the given condition

$$p(x) = x^3 - 3x.$$

14. Let $a_1, a_2, ..., a_{10}$ denote the weights of the 10 objects in decreasing order. It is given that $10 \geq a_1 \geq a_2 \geq ..... a_{10} \geq 1$ and that $a_1 + a_2 + ... + a_{10} = 20$. For each i, $1 \leq i \leq 9$, let $S_i = a_1 + ... + a_i$ (For example, $S_1 = a_1$, $S_2 = a_1 + a_2$, etc.). Consider the 11 numbers $0, S_1, S_2, ..., S_9, a_1 - a_{10}$. Note that all these 11 numbers are non-negative and we have $0 \leq a_1 - a_{10} < 10$ and $1 < S_i < 20$ for $1 \leq i \leq 9$. Now look at the remainders when these 11 numbers are divided by 10. We have 10 possible remainders and 11 numbers and hence by the pigeon-hole principle at least some two of these 11 numbers have the same remainder.

Case-I:

For some j, S_j has the remainder 0, *i.e.*, S_j is multiple of 10. But since $1 < S_j < 20$ the only possibility is that $S_j = 10$. Thus we get a balancing by taking the two groups to be $a_1, ..., a_j$ and $a_{j+1}, ..., a_{10}$.

Case-II:

Suppose $a_1 - a_{10}$ is a multiple of 10. But then since $0 \leq a_1 - a_{10} < 10$ this forces $a_1 - a_{10} = 10$ which in turn implies that all the weights are equal and equal to 2 as they add up to 20. In this case taking any five weights in one group and the remaining in the other we again get a balancing.

Case-III:

For some j and k, say $j < k$, we have that S_j and S_k have the same remainder, *i.e.*, $S_k - S_j$ is a multiple of 10. But again since $0 < S_k - S_j < 20$, we should have $S_k - S_j = 10$, *i.e.*, $a_k + a_{k-1} + ... + a_{j+1} = 10$ and we get a balancing.

Case-IV:

Suppose $(a_1 - a_{10})$ and S_j for some j $(1 \leq j \leq 9)$ have the same remainder, *i.e.*, $S_j(a_1 - a_{10})$is a multiple of 10. As in the previous cases this implies that

$$S_j - (a_1 - a_{10}) = 10$$

i.e., $a_2 + a_3 + ... + a_j + a_{10} = 10$

Therefore $\{a_2, a_3, ..., a_j, a_{10}\}$ and $\{a_1, a_{j+1}, ..., a_9\}$ balance each other.

Thus in all cases the given 10 objects can be split into two groups that balance each other.

ISI

Exam on : 10th May

PRACTICE PAPER

Indian Statistical Institute

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PART A

1. If $i^2 = -1$, then $(1 + i)^{20} - (1 - i)^{20}$ equals

(a) -1024 (b) $-1024i$

(c) 0 (d) 1024

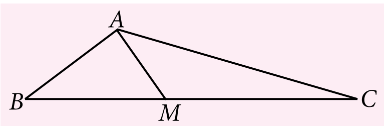
(e) $1024i$

2. A fair die is rolled six times. The probability of rolling at least five at least five times is

(a) $\frac{13}{729}$ (b) $\frac{12}{729}$ (c) $\frac{2}{729}$ (d) $\frac{3}{729}$

(e) None of these

3. In the adjoining figure triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?



(a) $2\sqrt{26}$ (b) $2\sqrt{31}$

(c) 9 (d) $4 + 2\sqrt{13}$

(e) Not enough information given to solve the problem

4. Ann and Barbara were comparing their ages and found that Barbara is as old as Ann was when Barbara was as old as Ann had been when Barbara was half as old as Ann is . If the sum of their present ages is 44 years, then Ann's age is

(a) 22 (b) 24 (c) 25 (d) 26

(e) 28

5. In a tennis tournament, n women and $2n$ men play, and each player plays exactly one match with every other player. If there are no ties and the ratio of the number of matches won by women to the number of matches won by men is 7/5, then n equals

(a) 2 (b) 4 (c) 6 (d) 7

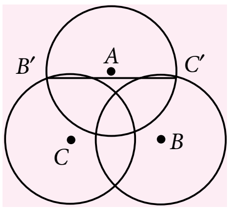
(e) None of these

6. An ordered pair (b, c) of integer, each of which has absolute value less than or equal to five, is chosen at random, with each such ordered pair having an equal likelihood of being chosen. What is the probability that the equation $x^2 + bx + c = 0$ will not have distinct positive real roots ?

(a) $\frac{106}{121}$ (b) $\frac{108}{121}$ (c) $\frac{110}{121}$ (d) $\frac{112}{121}$

(e) None of these

7. Circles with centers A, B and C each have radius r, where $1 < r < 2$. The distance between each pair of centers is 2. If B' is the point of intersection of circle A and circle C which is outside circle B, and if C' is the point of intersection of circle A and circle B which is outside circle C, then length $B'C'$ equals



(a) $3r - 2$ (b) r^2

(c) $r+\sqrt{3(r-1)}$ (d) $1+\sqrt{3(r^2-1)}$

(e) None of these

8. A box contains 2 pennies, 4 nickels and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What

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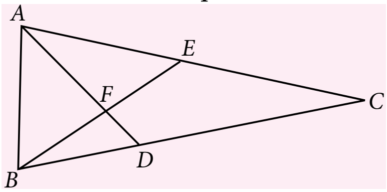
is the probability that the value of the coins drawn is at least 50 cents?

(a) $\frac{37}{924}$ (b) $\frac{91}{924}$ (c) $\frac{127}{924}$ (d) $\frac{132}{924}$

(e) None of these

9. In triangle ABC, $\angle CBA = 72°$, E is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. The ratio of the area of ΔBDF to the area of quadrilateral $FDCE$ is



(a) 1/5 (b) 1/4 (c) 1/3 (d) 2/5

(e) None of these

10. A six digit number (base 10) is squarish if it satisfies the following conditions:

(i) None of its digits is zero;

(ii) It is a perfect square; and

(iii) The first two digits, the middle two digits and the last two digits of the number are all perfect squares when considered as two digit numbers.

How many squarish numbers are there?

(a) 0 (b) 2 (c) 3 (d) 8

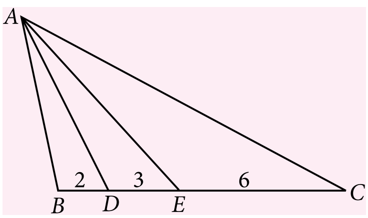
(e) 9

11. How many lines in a three-dimensional rectangular coordinate system pass through four distinct points of the form (i, j, k), where i, j and k are positive integers not exceeding four?

(a) 60 (b) 64 (c) 72 (d) 76

(e) 100

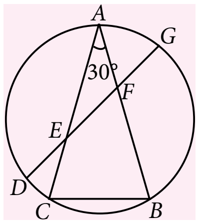
12. In triangle ABC in the adjoining figure, AD and AE trisect $\angle BAC$. The lengths of BD, DE and EC are 2, 3, and 6, respectively. The length of the shortest side of ΔABC is



(a) $2\sqrt{10}$ (b) 11 (c) $6\sqrt{6}$ (d) 6

(e) Not uniquely determined by the given information

13. In the adjoining figure triangle ABC is inscribed in a circle. Point D lies on $\overset{\frown}{AC}$ with $\overset{\frown}{DC} = 30°$, and point G lies on $\overset{\frown}{BA}$ with $\overset{\frown}{BG} > \overset{\frown}{GA}$. Side AB and side AC each have length equal to the length of chord DG, and $\angle CAB = 30°$. Chord DG intersects sides AC and AB at E and F, respectively. The ratio of the area of ΔAFE to the area of ΔABC is



(a) $\frac{2-\sqrt{3}}{3}$ (b) $\frac{2\sqrt{3}-3}{3}$

(c) $7\sqrt{3}-12$ (d) $3\sqrt{3}-5$

(e) $\frac{9-5\sqrt{3}}{3}$

14. Consider the set of all equations :
$x^3 + a_2x^2 + a_1x + a_0 = 0$, where a_2, a_1, a_0 are real constants and $|a_i| \le 2$ for $i = 0, 1, 2$. Let r be the largest positive real number which satisfies at least one of these equations. Then

(a) $1 \le r < \frac{3}{2}$ (b) $\frac{3}{2} \le r < 2$

(c) $2 \le r < \frac{5}{2}$ (d) $\frac{5}{2} \le r < 3$

(e) $3 \le r < \frac{7}{2}$

15. The lengths of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is

(a) 3/4 (b) 7/10 (c) 2/3 (d) 9/14

(e) None of these

16. Suppose $z = a + ib$ is a solution of the polynomial equation $c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0 = 0$, where c_0, c_1, c_2, c_3, c_4, a and b are real constants and $i^2 = -1$. Which one of the following must also be a solution ?

(a) $-a - ib$ (b) $a - ib$

(c) $-a + ib$ (d) $b + ia$

(e) None of these

17. A set of consecutive positive integers beginning with 1 is written on a blackboard. One number is erased. The average (arithmetic mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

(a) 6 (b) 7 (c) 8 (d) 9

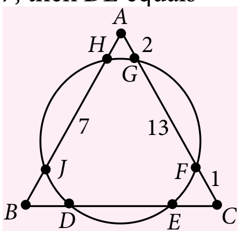
(e) Can not be determined

18. Let x, y and z be three positive real numbers whose sum is 1. If no one of these numbers is more than twice any other, then the minimum possible value of the product xyz is

(a) $\frac{1}{32}$ (b) $\frac{1}{36}$ (c) $\frac{4}{125}$ (d) $\frac{1}{127}$

(e) None of these

19. In the adjoining figure, the circle meets the sides of an equilateral triangle at six points. If $AG = 2$, $GF = 13$, $FC = 1$ and $HJ = 7$, then DE equals



(a) $2\sqrt{22}$ (b) $7\sqrt{3}$ (c) 9 (d) 10

(e) 13

PART B

1. If a, b, c are the sides of a triangle and $a + b + c = 2$, prove that $a^2 + b^2 + c^2 + 2abc < 2$.

2. For any $x \in R$ and $n \in N$, define

$x^{(0)} = 1$ and $x^{(n)} = x(x - 1) \ldots. (x - n + 1)$.

Show that $(x+y)^{(n)} = \sum_{k=0}^{n} \binom{x}{k} x^{(k)} y^{(n-k)}$

3. Find all real x, y satisfying $x^3 + y^3 = 7$ and $x^2 + y^2 + x + y + xy = 4$.

4. Determine the set of all positive integers n for which 3^{n+1} divides $2^{3^n} + 1$. Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

5. Prove that the product of 4 consecutive natural numbers cannot be a cube.

6. Suppose $ABCD$ is a convex quadrilateral and P, Q are the midpoints of CD, AB. Let AP, DQ meet in X and BP, CQ meet in Y. Prove that $[ADX] + [BCY] = [PXQY]$. How does the conclusion alter if $ABCD$ is not a convex quadrilateral ?

7. Triangle ABC has incentre I and the incircle touches BC, CA at D, E respectively . If BI meets DE in G, show that AG is perpendicular to BG.

8. Suppose $ABCD$ is a quadrilateral such that a semicircle with its centre at the midpoint of AB and bounding diameter lying on AB touches the other three sides BC, CD and DA. Show that $AB^2 = 4BC \cdot AD$.

9. Suppose $ABCD$ is a rectangle and P, Q, R, S are points on the sides AB, BC, CD, DA respectively. Show that $PQ + QR + RS + SP \geq \sqrt{2}\ AC$.

10. How many increasing 3-term geometric progressions can be obtained from the sequence 1, 2, 2^2, 2^3, ..., 2^n ? (*e.g.*, $\{2^2, 2^5, 2^8\}$ is a 3-term geometric progression for $n \geq 8$.)

11. For any natural number n, ($n \geq 3$), let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (*e.g.*, $f(3) = 1, f(4) = 0, f(7) = 2$). Show that

(i) $f(1999) > f(1996)$

(ii) $f(2000) = f(1997)$

SOLUTION

PART A

1. (c) : 1st Solution: Since $i^2 = -1$, $(1 + i)^2 = 2i$ and $(1 - i)^2 = -2i$. Writing

$$(1 + i)^{20} - (1 - i)^{20} = ((1 + i)^2)^{10} - ((1 - i)^2)^{10}$$

we have,

$$(1 + i)^{20} - (1 - i)^{20} = (2i)^{10} - (-2i)^{10} = 0$$

Observe that

$$i(1 - i) = i - i^2 = 1 + i, \text{ and } i^4 = (-1)^2 = 1$$

Therefore, $(1 + i)^4 = i^4(1 - i)^4 = (1 - i)^4$,

and $(1 + i)^{4n} - (1 - i)^{4n} = 0$ for $n = 1, 2, 3, \ldots.$

2nd Solution:

$$(1+i)^{20} = (\sqrt{2})^{20}[\cos 900° + i \sin 900°] = -1024,$$

$$(1-i)^{20} = (\sqrt{2})^{20}[\cos(-900°) + i \sin(-900°)] = -1024$$

and the difference is 0.

2. (a) : Let A be the event of rolling at least five; then the probability of A is $\frac{2}{6} = \frac{1}{3}$. In six rolls of a die the probability of event A happening six times is $\left(\frac{1}{3}\right)^6$. The probability of getting exactly five successes and one failure of A in a specific order is $\left(\frac{1}{3}\right)^5 \cdot \frac{2}{3}$. Since there are six ways to do this, the probability of getting five successes and one failure of A in any order is

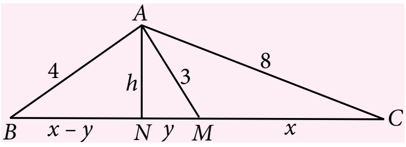
$$6\left(\frac{1}{3}\right)^5 \cdot \frac{2}{3} = \frac{12}{729}.$$

The probability of getting all successes or five successes and one failure in any order is thus

$$\frac{12}{729} + \left(\frac{1}{3}\right)^6 = \frac{13}{729}.$$

3. (b) : 1st Solution: In the adjoining figure, let h be the length of altitude AN drawn to BC, let $x = BM$ and let $y = NM$. Then

$$h^2 + (x + y)^2 = 64,$$
$$h^2 + y^2 = 9$$
$$h^2 + (x - y)^2 = 16.$$



Subtracting twice the second equation from the sum of the first and third equations yield $2x^2 = 62$. Thus $x = \sqrt{31}$ and $BC = 2\sqrt{31}$.

2nd Solution: Recall that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals. Applying this to the parallelogram having AB and AC as adjacent sides yield $2(4^2 + 8^2) = 6^2 + (2x)^2$, $x = \sqrt{31}$.

4. (b) : The table below shows the ages of Ann and Barbara at various times referred to in the problem. The first column indicates their present ages. The second column shows their ages when Barbara was half as old as Ann is now – that was $y - \left(\frac{x}{2}\right)$ years ago, hence Ann's age was $x - \left(y - \frac{x}{2}\right)$ or $\frac{3x}{2} - y$. The third column refers to the time when Barbara was as old as Ann had been when Barbara was half as old as Ann is that was $y - \left(\frac{3x}{2} - y\right)$ or $2y - \frac{3x}{2}$ years ago; hence Ann's age then was $x - \left(2y - \frac{3x}{2}\right)$ or $\frac{5x}{2} - 2y$.

Ann	x	$\frac{3x}{2} - y$	$\frac{5x}{2} - 2y$
Barbara	y	$\frac{x}{2}$	$\frac{3x}{2} - y$

By the conditions stated in the problem, $x + y = 44$ and $y = \frac{5x}{2} - 2y$; the simultaneous solution of these yield $x = 24$.

5. (c) : 1st Solution: Let $7m$, $5m$ be the total number of matches won by women and men, respectively. Now there are $\frac{n(n-1)}{2}$ matches between women, hence won by women.

There are $\frac{2n(2n-1)}{2} = n(2n-1)$ matches between men, hence won by men. Finally, there are $2n \cdot n = 2n^2$ mixed matches, of which $k = 7m - \frac{n(n-1)}{2}$ are won by women, and $2n^2 - k = 5m - n(2n - 1)$ by men.

We note for later use that $k \le 2n^2$. Then

$$\frac{7m}{5m} = \frac{\frac{1}{2}n(n-1)+k}{n(2n-1)+2n^2-k} = \frac{1}{2}\frac{n(n-1)+2k}{4n^2-n-k}$$

$\Rightarrow \quad 5n(n - 1) + 10k = 14(4n^2 - n - k)$

$\Rightarrow \quad 51n^2 - 9n - 24k = 3(17n^2 - 3n - 8k) = 0,$

$\Rightarrow \quad 17n^2 - 3n = 8k, \ \frac{17n^2 - 3n}{8} = k.$

Since $k \le 2n^2$, it follows that $17^2 - 3n \le 16n^2$, so $n(n - 3) \le 0$, $n \le 3$. Since k is an integer, $n \ne 1$ or 2; hence $n = 3$.

2nd Solution: The total number of games played was

$$\binom{3n}{2} = \frac{3n(3n-1)}{2}$$

Suppose that $7m$ games were won by women, and $5m$ games were won by men. Then

$7m + 5m = 12m = \frac{3n(3n-1)}{2}$ and $3n(3n - 1) = 24m$.

Since m is an integer, $3n(3n - 1)$ is a multiple of 24. But this is not true for $n = 2, 4, 6, 7$. (It is true for $n = 3$.)

6. (e) : There are six statements equivalent for integers b, c with absolute value at most 5:

1) the equation $x^2 + bx + c = 0$ has positive roots;
2) the equation $x^2 + bx + c = 0$ has real roots, the smaller of which is positive;
3) $\sqrt{b^2 - 4c}$ is real and $-b - \sqrt{b^2 - 4c} > 0$;
4) $0 \le b^2 - 4c < b^2$ and $b < 0$;
5) $0 < c \le \frac{b^2}{4}$ and $b < 0$;
6) $b = -2$ and $c = 1$; or $b = -3$ and $c = 1$ or 2; or $b = -4$ and $c = 1, 2, 3$ or 4; or $b = -5$ and $c = 1, 2, 3, 4$ or 5.

The roots corresponding to the pairs (b, c) described in (6) will be distinct unless $b^2 = 4c$. Thus, deleting $(b, c) = (-2, 1)$ and $(-4, 4)$ from the list in (6) yields the ten pairs resulting in distinct positive roots.

The desired probability is then $1 - \frac{10}{11^2} = \frac{111}{121}$.

7. (d) : Denote by A' the analogous intersection point of the circles with centers at B and C, so that, by symmetry, $A'B'C'$ and ABC are both equilateral triangles. Again, by symmetry, ΔABC and $\Delta A'B'C'$ have a common centroid; call it K. Let M be the midpoint of the line segment BC. From the triangle $A'BC$ we see that the length $A'M = \sqrt{r^2-1}$. Since corresponding lengths in similar triangles are proportional,

$$\frac{B'C'}{BC} = \frac{A'K}{AK}.$$

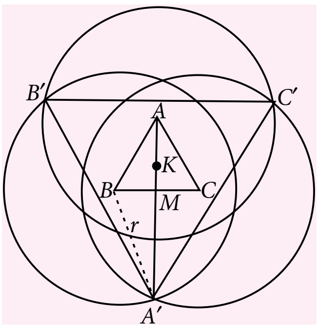
Since equilateral ΔABC has sides of length 2, we find that $B'C' = 2\frac{A'K}{AK}$ and also that altitude AM has length $\sqrt{3}$.

Consequently,

$$AK = \frac{2}{3}AM = \frac{2}{3}\sqrt{3},\ MK = \frac{1}{3}AM = \frac{1}{3}\sqrt{3}$$

and $A'K = A'M + MK = \sqrt{r^2-1} + \left(\frac{\sqrt{3}}{3}\right)$

Thus, $$B'C' = 2\frac{\sqrt{r^2-1}+\left(\frac{\sqrt{3}}{3}\right)}{2\left(\frac{\sqrt{3}}{3}\right)} = \sqrt{3(r^2-1)}+1.$$



8. (c) : The number of ways of choosing 6 coins from 12 is $\binom{12}{6} = 924$

[The symbol $\binom{n}{k}$ denotes the number of ways k things may be selected from a set of n distinct objects.] "Having at least 50 cents" will occur if one of the following cases occurs:

1) Six dimes are drawn.
2) Five dimes and any other coin are drawn.
3) Four dimes and two nickels are drawn.

The number of ways 1), 2) and 3) can occur are $\binom{6}{6}$, $\binom{6}{5}\binom{6}{1}$ and $\binom{6}{4}\binom{4}{2}$, respectively. The desired probability is, therefore,

$$\frac{\binom{6}{6}+\binom{6}{4}\binom{4}{2}+\binom{6}{5}\binom{6}{1}}{924} = \frac{127}{924}.$$

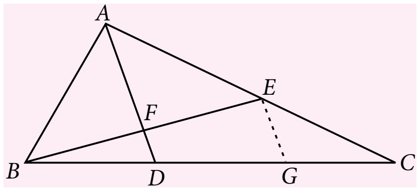
9. (a) : In the adjoining figure the line segment from E to G, the midpoint of DC, is drawn. Then

area $\Delta EBG = \left(\frac{2}{3}\right)$(area ΔEBC)

area $\Delta BDF = \left(\frac{1}{4}\right)$(area ΔEBG) $= \left(\frac{1}{6}\right)$(area ΔEBC)

(Note that since EG connects the midpoints of sides AC and DC in ΔACD, EG is parallel to AD.) Therefore,

area $FDCE = \left(\frac{5}{6}\right)$(area ΔEBC)



and $\dfrac{\text{area } \Delta BDF}{\text{area } FDCE} = \dfrac{1}{5}$.

The measure of $\angle CBA$ was not needed.

10. (b) : That N is squarish may be expressed algebraically as follows: there are single digit integers A, B, C, a, b, c such that

$$N = 10^4A^2 + 10^2B^2 + C^2 = (10^2a + 10b + c)^2,$$

where each of A, B, C exceeds 3, and so a and c are positive. Since $10^2B^2 + C^2 < 10^4$, we can write

$$10^4A^2 < (10^2a + 10b + c)^2 < 10^4A^2 + 10^4 < 10^4(A + 1)^2$$

Taking square roots we obtain

$$100A < 100a + 10b + c < 100A + 100,$$

from which it follows that $A = a$. Hence $a \geq 4$. Now consider

$$M = N - 10^4A^2 = (10^2a + 10b + c)^2 - 10^4a^2$$
$$= 10^3(2ab) + 10^2(b^2 + 2ac) + 10(2bc) + c^2.$$

Since M has only four digits, $2ab < 10$, which implies that $ab \leq 4$. Thus either (i) $b = 0$, or (ii) $a = 4$ and $b = 1$. In case (ii),

$$N = (410 + c)^2 = 168100 + 820c + c^2$$

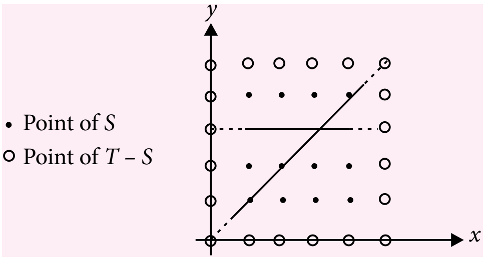
If $c = 1$ or 2, the middle two digits of N form a number exceeding 81, hence not a square. If $c \geq 3$, then the leftmost two digits of N are 17. Therefore case (i) must hold, and we have

$$N = (10^2a + c)^2 = 10^4a^2 + 10^2(2ac) + c^2$$

Thus $a \geq 4$, $c \geq 4$ and $2ac$ is an even two-digit perfect square. It is now easy to check that either $a = 8$, $c = 4$, $N = 646416$, or $a = 4$, $c = 8$, $N = 166464$.

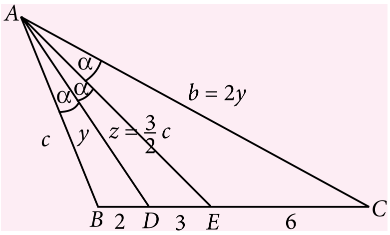
11. (d) : 1st Solution: Consider the smallest cube containing all the lattice points (i, j, k), $1 \le i, j, k \le 4$, in a three dimensional Cartesian coordinate system. There are 4 main diagonals. There are 24 diagonal lines parallel to a coordinate plane: 2 in each of four planes parallel to each of the three coordinate planes. There are 48 lines parallel to a coordinate axis: 16 in each of the three directions. Threrfore, there are
$4 + 24 + 48 = 76$ lines.

2nd Solution: Let S be the set of lattice points (i, j, k) with $1 \le i, j, k \le 4$, and let T be the set of lattice points (i, j, k) with $0 \le i, j, k \le 5$. Every line segment containing four points of S can be extended at both ends so as to contain six points of T.



12. (a) : 1st Solution: In the adjoining figure let $\angle BAC = 3\alpha$, $c = AB$, $y = AD$, $z = AE$ and $b = AC$. Then by the angle bisector theorem

1) $\frac{c}{z} = \frac{2}{3}$ and $\frac{y}{b} = \frac{1}{2}$; so $z = \frac{3}{2}c$, $b = 2y$.



Using the law of cosines in ΔADB, ΔAED and ΔACE, respectively, yields the following expressions for $\cos\alpha$.

$$\frac{c^2 + y^2 - 4}{2cy} = \frac{\frac{9}{4}c^2 + y^2 - 9}{3cy} = \frac{\frac{9}{4}c^2 + 4y^2 - 36}{6cy}$$

The equality of the first and second expressions implies $3c^2 - 2y^2 = 12$.
The equality of the first and third expressions implies $3c^2 - 4y^2 = -96$.
Solving these two equations for c^2 and y^2 yields

$c^2 = 40$, $y^2 = 54$

Thus the sides of the triangle are

$AB = c = 2\sqrt{10} \approx 6.3$

$AC = b = 2y = 2\sqrt{54} = 6\sqrt{6} \approx 14.7$,

$BC = 11$

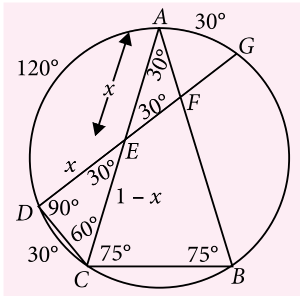
2nd Solution: Using the angle bisector formula and (1) above, we have

$$y^2 + 6 = cz = \frac{2}{3}z^2, \; z^2 + 18 = yb = 2y^2$$

Solving these equations for y^2 and z^2 yields $y^2 = 54$, $z^2 = 90$. Therefore,

$$AB = c = \frac{2}{3}\sqrt{90} = 2\sqrt{10},\; AC = b = 2\sqrt{54} = 6\sqrt{6}.$$

13. (c) : In the figure, line segment DC is drawn. Since $\overset{\frown}{AC} = 150°$, $\overset{\frown}{AD} = \overset{\frown}{AC} - \overset{\frown}{DC} = 150° - 30° = 120°$. Hence $\angle ACD = 60°$. Since $AC = DG$, $\overset{\frown}{GA} = \overset{\frown}{GD} - \overset{\frown}{AD} = \overset{\frown}{AC} - 120° = 30°$. Therefore $\overset{\frown}{CG} = 180°$ and $\angle CDG = 90°$. So ΔDEC is a $30° - 60° - 90°$ triangle.



Since we are looking for the ratio of the areas, let us assume without loss of generality that $AC = AB = DG = 1$. Since AC and DG are chords of equal length in a circle, we have $AE = DE$. Let x be their common length. Then $CE = 1 - x = \frac{2x}{\sqrt{3}}$. Solving for x yields $AE = x = 2\sqrt{3} - 3$. Since BA and DG are chords of equal length in a circle, we have $FG = FA$, and since ΔFAE is isosceles, $EF = FA$. Thus

$$EF = FG = \frac{1}{2}(1 - x)$$

Therefore,

$$\text{area } \Delta AFE = \frac{1}{2}(AE)(AF)\sin 30°$$

$$= \frac{1}{2}x\frac{1-x}{2}.\frac{1}{2} = \frac{x - x^2}{8} = \frac{7\sqrt{3} - 12}{4}$$

$$\text{area } \Delta ABC = \frac{1}{2}(AB)(AC)\sin 30° = \frac{1}{4}$$

Hence, $\frac{\text{area } \Delta AFE}{\text{area } \Delta ABC} = 7\sqrt{3} - 12$

14. (d) : Let $g(x) = x^3 + a_2x^2 + a_1x + a_0$ be an arbitrary cubic with constants of the specified form. Because x^3 dominates the other terms for large enough x,

$g(x) > 0$ for all x greater than the largest real root of g. Thus we seek a particular g in which the terms $a_2x^2 + a_1x + a_0$ "hold down" $g(x)$ as much as possible, so that the value of the largest real root is as large as possible. This suggests that the answer to the problem is the largest root of $f(x) = x^3 - 2x^2 - 2x - 2$. Call this root r_0. Since $f(0) = -2$, r_0 is certainly positive. To verify this conjecture, note that for $x \geq 0$,

$$-2x^2 \leq a_2x^2,\ -2x \leq a_1x, \text{ and } -2 \leq a_0.$$

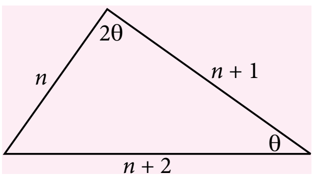
Summing these inequalities and adding x^3 to both sides yields $f(x) \leq g(x)$ for all $x \geq 0$. Thus for all $x > r_0$, $0 < f(x) \leq g(x)$. That is, no g has a root larger than r_0, so r_0 is the r of the problem.

A sketch of f shows that it has a typical cubic shape, with largest root a little less than 3. In fact, $f(2) = -6$ and $f(3) = 1$. To be absolutely sure the answer is (d), not (c), compute $f\left(\frac{5}{2}\right)$ to see if it is negative. Indeed,

$$f\left(\frac{5}{2}\right) = -\frac{31}{8}.$$

15. (a) : In the adjoining figure, n denotes the length of the shortest side, and θ denotes the measure of the smallest angle. Using the law of sines and writing $2\sin\theta\cos\theta$ for $\sin 2\theta$, we obtain

$$\frac{\sin\theta}{n} = \frac{2\sin\theta\cos\theta}{n+2},\quad \cos\theta = \frac{n+2}{2n}$$



Equating $\frac{n+2}{2n}$ to the expression of $\cos\theta$ obtained from the law of consines yields

$$\frac{n+2}{2n} = \frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)}$$

$$= \frac{(n+1)(n+5)}{2(n+1)(n+2)} = \frac{n+5}{2(n+2)}$$

Thus $n = 4$ and $\cos\theta = \frac{4+2}{4(2)} = \frac{3}{4}$

16. (c) : We recall the theorem that complex roots of polynomials with real coefficients come in conjugate pairs. Though not applicable to the given polynomial, that theorem is proved by a technique which we can use to work this problem too. Namely, conjugate both sides of the original equation

$$0 = c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0$$

obtaining

$$0 = c_4\bar{z}^4 - ic_3\bar{z}^3 + c_2\bar{z}^2 - ic_1\bar{z} + c_0$$

$$= c_4(-\bar{z})^4 + ic_3(-\bar{z})^3 + c_2(-\bar{z})^2 - ic_1(-\bar{z}) + c_0$$

That is, $-\bar{z} = -a + ib$ is also a solution of the original equation. (One may check by example that neither $-a - ib$ nor $a - ib$ nor $b + ia$ need be a solution.) For instance, consider the equation $z^4 - iz^3 = 0$ and the solution $z = i$. Here $a = 0$, $b = 1$. Neither $-i$ nor 1 is a solution. [Alternatively, the substitution $z = i\omega$ into the given equation makes the coefficients real and the above quoted theorem applicable.]

17. (b) : Let n be the last number on the board. Now the largest average possible is attained if 1 is erased; the average is then

$$\frac{2+3+\ldots+n}{n-1} = \frac{\frac{(n+1)n}{2} - 1}{n-1} = \frac{n+2}{2}$$

The smallest average possible is attained when n is erased; the average is then

$$\frac{n(n-1)}{2(n-1)} = \frac{n}{2}$$

Thus, $\frac{n}{2} \leq 35\frac{7}{17} \leq \frac{n+2}{2}$, $n \leq 70\frac{14}{17} \leq n+2$,

$$68\frac{14}{17} \leq n \leq 70\frac{14}{17}.$$

Hence $n = 69$ or 70. Since $35\frac{7}{17}$ is the average of $(n - 1)$ integers, $\left(35\frac{7}{17}\right)(n-1)$ must be an integer and n is 69. If x is the number erased, then

$$\frac{\frac{1}{2}(69)(70) - x}{68} = 35\frac{7}{17}$$

So $69 \cdot 35 - x = \left(35\frac{7}{17}\right)68 = 35 \cdot 68 + 28$

$$35 - x = 28 \quad \text{or} \quad x = 7.$$

18. (a) : Let $m = x_0y_0z_0$ be the minimum value, and label the numbers so that $x_0 \leq y_0 \leq z_0$. In fact $z_0 = 2x_0$, for if $z_0 < 2x_0$, then by decreasing x_0 slightly, increasing z_0 by the same amount, and keeping y_0 fixed, we would get new values which still meet the constraints but which have a smaller product–contradiction! To show this contradiction formally, let $x_1 = x_0 - h$ and $z_1 = z_0 + h$, where $h > 0$ is so small that $z_1 \leq 2x_1$ also. Then x_1, y_0, z_1 also meet all the original constraints, and

$$x_1y_0z_1 = (x_0 - h)y_0(z_0 + h)$$

$$= x_0y_0z_0 + y_0[h(x_0 - z_0) - h^2] < x_0y_0z_0$$

So $z_0 = 2x_0$, $y_0 = 1 - x_0 - z_0 = 1 - 3x_0$, and

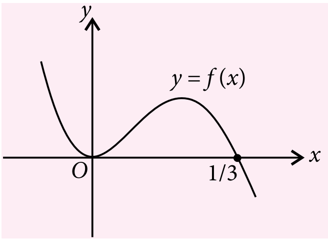
$$m = 2x_0^2(1 - 3x_0)$$

Also, $x_0 \le 1 - 3x_0 \le 2x_0$, or equivalently, $\frac{1}{5} \le x_0 \le \frac{1}{4}$.

Thus m may be viewed as a value of the function

$$f(x) = 2x^2(1 - 3x)$$

On the domain $D = \left\{x \middle| \frac{1}{5} \le x \le \frac{1}{4}\right\}$. In fact, m is the smallest value of f on D, because minimizing f on D is just a restricted version of the original problem: for each $x \in D$, setting $y = 1 - 3x$ and $z = 2x$ gives x, y, z meeting the original constraints, and makes $f(x) = xyz$.

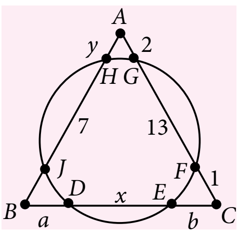


To minimize f on D, first sketch f for all real x. (See Figure.) Since f has a relative minimum at $x = 0$ ($f(x)$ has the same sign as x^2 for $x < \frac{1}{3}$), and cubics have at most one relative minimum, the minimum of f on D must be at one of the end points. In fact,

$$f\left(\frac{1}{4}\right) = \frac{1}{32} \le f\left(\frac{1}{5}\right) = \frac{4}{125}$$

19. (a) : In the adjoining figure, let $AH = y$, $BD = a$, $DE = x$ and $EC = b$. We are given $AG = 2$, $GF = 13$, $HJ = 7$ and $FC = 1$. Thus the length of the side of the equilateral triangle is 16. Also, using the theorem about secants drawn to a circle from an external point, we have $y(y + 7) = 2(2 + 13)$, or

$$0 = y^2 + 7y - 30 = (y - 3)(y + 10)$$



Hence $y = 3$ and $BJ = 6$. Using the same theorem we have $b(b + x) = 1(1 + 13) = 14$ and

$$a(a + x) = 6(6 + 7) = 78.$$

Also, $a + b + x = 16$.

There are many ways of solving the system

(1) $a^2 + ax = 78$,

(2) $b^2 + bx = 14$,

(3) $a + b + x = 16$.

The one given below allows us to find x without first having to find a and b.

Subtract the second equation from the first, factor out $(a - b)$, and use (3):

$$a^2 - b^2 + (a - b)x = (a - b)(a + b + x)$$
$$= (a - b)16 = 78 - 14 = 64.$$

Therefore, $a - b = 4$.

Adding this to (3) we obtain $2a + x = 20$, whence

$$a = 10 - \left(\frac{x}{2}\right)$$

This, substituted into (1), yields

$$\left(10 - \frac{x}{2}\right)\left[10 - \frac{x}{2} + x\right] = \left(10 - \frac{x}{2}\right)\left(10 + \frac{x}{2}\right)$$
$$= 100 - \frac{x^2}{4} = 78$$

$$\frac{x^2}{4} = 22 \quad \Rightarrow \quad x = 2\sqrt{22}$$

PART B

1. Since $b + c > a \Rightarrow a + b + c > 2a$, $1 - a$ is positive; similarly $1 - b$ and $1 - c$ are positive. Therefore

$$(1 - a)(1 - b)(1 - c) > 0$$

i.e., $1 - (a + b + c) + ab + bc + ca - abc > 0$

This yields $-2(ab + bc + ca) + 2abc < -2$

We also have $a^2 + b^2 + c^2 + 2(ab + bc + ca) = 4$

Adding, we get the result.

2. **1st Solution:** Proof by induction : Result is clearly true for $n = 1$. Assume the relation to be true for all $n \le m$. To prove the result for $n = m + 1$

$$\sum_{k=0}^{m+1} \binom{m+1}{k} x^{(k)} y^{(m+1-k)}$$

$$= \sum_{k=0}^{m+1} \left[\binom{m}{k} + \binom{m}{k-1}\right] x^{(k)} y^{(m+1-k)}$$

$$\left(\text{Note} : \binom{m}{m+1} = \binom{m}{-1} = 0\right)$$

$$= \sum_{k=0}^{m} \binom{m}{k} x^{(k)} (y-1)^{(m-k)} y$$
$$+ \sum_{k=0}^{m+1} \binom{m}{k-1} x \cdot (x-1)^{(k-1)} y^{(m+1-k)}$$

$$= y \sum_{k=0}^{m} \binom{m}{k} x^{(k)} (y-1)^{(m-k)} + x \sum_{l=0}^{m} \binom{m}{l} (x-1)^{(l)} y^{(m-l)}$$

$= y(x + y - 1)^{(m)} + x(x - 1 + y)^{(m)}$

(by induction hypothesis)

$= (x + y)(x + y - 1)^{(m)}$
$= (x + y)^{(m - 1)}$

2nd Solution: By using Leibnitz's theorem for successive differentiation. If u and v are two functions of a variable t, each being differentiable n times then uv also has n^{th} derivative given as follows :

$$\frac{d^n(uv)}{dt^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k u}{dt^k} \cdot \frac{d^{n-k} v}{dt^{n-k}}.$$

Let $u = t^x$, $v = t^y$. Then L.H.S. is

$$(x + y)(x + y - 1) \ldots (x + y - n + 1)t^{x + y - n}$$
$$(x + y)^{(n)}\, t^{x + y - n}$$

R.H.S. is

$$\sum_{k=0}^{n} \binom{n}{k} x(x-1)\ldots(x-k+1)\, t^{x-k} \times y(y-1) \ldots (y - (n - k) + 1)t^{y - (n - k)}$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)} \right) t^{x+y-n}$$

So cancelling $t^{x + y - n}$ we get the required result.

3. Let $\alpha = x + y$ and $\beta = xy$. Then

$$x^2 + y^2 = \alpha^2 - 2\beta \text{ and } x^3 + y^3 = \alpha^3 - 3\alpha\beta$$

Therefore the given equations can be written as

$$\alpha^3 - 3\alpha\beta = 7 \text{ and } \alpha^3 + \alpha - \beta = 4$$

Also note that α is positive, since

$$\frac{1}{2}(x+y)[(x-y)^2 + x^2 + y^2] = x^2 + y^2 = 7$$

Now $\beta = \alpha^2 + \alpha - 4$ and $\alpha^3 - 3\alpha(\alpha^2 + \alpha - 4) = 7$

$\Rightarrow \quad 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0$

$\Rightarrow \quad (\alpha - 1)^2 (2\alpha + 7) = 0$

$\Rightarrow \quad \alpha = 1 \quad (\because \alpha > 0)$

So, $\beta = \alpha^2 + \alpha - 4 = -2$

i.e., $x + y = 1$ and $xy = -2$.

Solving we get $(x, y) = (2, -1)$ or $(-1, 2)$.

4. For any $m \in Z$, let $\theta(m)$ denote the highest power of 3 which divides m. Let $t_n = 2^{3^n} + 1$ where $n \in N$. Then for any $n \in N$,

$$t_{n+1}(2^{3^n})^3 + 1 = (t_n - 1)^3 + 1$$

Which by simplification gives $t_{n+1} = t_n^3 - 3t_n^2 + 3t_n$. Now let us show by induction $\theta(t_n) = 3^{n+1}$. When $n = 1$, it is obvious. Assume $n \geq 1$ such that $\theta(t_n) = 3^{n+1}$. Then

$$\theta(t_{n+1}) = \theta(t_n^3 - 3t_n^2 + 3t_n)$$
$$= \theta(t_n)\, \theta\, (t_n^2 - 3t_n + 3) = 3^{n+1} \cdot 3 = 3^{n+2}$$

5. Consider the product $n(n + 1)(n + 2)(n + 3)$ of four consecutive numbers. Suppose $n > 1$ as $1 \cdot 2 \cdot 3 \cdot 4 = 24$ is anyway not a cube. We use the fact that if the product of two relatively prime numbers is a cube then each of the two numbers is itself a cube.

Case-I: Suppose n is even. Then $(n + 1)$ is relatively prime to $n(n + 2)(n + 3)$. Thus if $n(n + 1)(n + 2)(n + 3)$ is a cube we must have that $(n + 1)$ and $n(n + 1)(n + 3)$ are cubes. But $n(n + 2)(n + 3)$ is not a cube since it lies strictly between two consecutive cubes:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1 < n(n + 2)(n + 3)$$

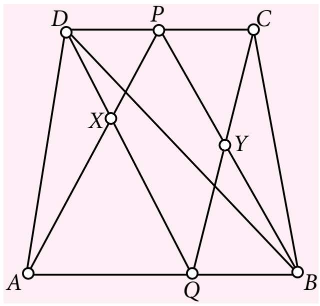
and $n(n + 2)(n + 3) < (n + 2)^3 = n^3 + 6n^2 + 12n + 8$

Thus, $n(n + 1)(n + 2)(n + 3)$ is not a cube for even n.

Case-II: Suppose n is odd. The proof is similar by noting that this time

$$n(n + 1)(n + 3) \text{ is prime to } (n + 2).$$

6. We denote areas of triangles ABC, quadrilaterals $ABCD$, etc. by $[ABC]$, $[ABCD]$ etc. Join PQ and draw one of the diagonals, say BD. We use the fact that the median of a triangle bisects its area.



From triangles DAB (with median DQ) and BCD (with median BP), we have

$$[ADQ] = [BDQ] \text{ and } [BPC] = [BPD].$$

Adding, we have

$$[ADQ] + [BPC] = [BDQ] + [BPD]$$
$$= [BPDQ] = [BPQ] + [DPQ]$$
$$= [APQ] + [CPQ],$$

Since PQ is a median of both the triangles APB and CQD. Writing in terms of smaller areas, we have

$$[AXQ] + [AXD] + [BYC] + [PYC]$$
$$= [AXQ] + [PXQ] + [CPY] + [QPY].$$

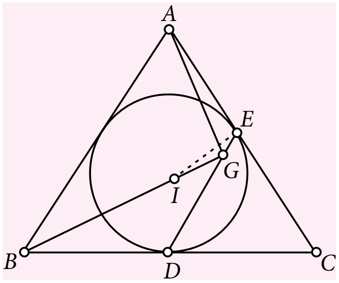
On cancellation, this yields,

$$[ADX] + [BCY] = [PXQY].$$

If $ABCD$ is a concave quadrilateral and the points P, Q, X, Y are located as in the problem, then by a similar argument, we arrive at the relation $|[ADX] - [BCY]| = [PXQY]$, where the left hand side denotes the modulus of the difference of areas.

7. **1st Solution:** Join ID and AI. We first show that triangles BDG and BIA are similar. We have

$$\angle DBG = \frac{B}{2} = \angle IBA \qquad \ldots(1)$$



Further ΔCDE is isosceles, because $CD = CE$. So

$$\angle CDE = \angle CED = 90^\circ - \frac{C}{2}.$$

Therefore $\angle BDG = 180^\circ - \angle CDE = 90^\circ + \dfrac{C}{2}$.

Also, $\angle BIA = 180^\circ - \angle ABI - \angle BAI$

$$= 180^\circ - \frac{B}{2} - \frac{A}{2} = 90^\circ + \frac{C}{2}$$

Thus, $\angle BDG = \angle BIA$ ....(2)

From (1) and (2), it follows that, ΔBDG is similar to ΔBIA. Therefore $\dfrac{BD}{BI} = \dfrac{BG}{BA}$.

This fact along with the relation $\angle DBI = \angle GBA \left(= \dfrac{B}{2}\right)$ implies that triangles DBI and GBA are similar. Consequently, $\angle BDI = \angle BGA$. But $\angle BDI = 90^\circ$. So $\angle BGA = 90^\circ$. That is, BG is perpendicular to AG.

2nd Solution: Join IE. We show that $IGEA$ is a cyclic quadrilateral. As in the previous method,

$$\angle CED = \angle CDE = 90^\circ - \frac{C}{2}.$$

$$\text{Therefore, } \angle AEG = 180^\circ - \angle DEC = 180^\circ - \left(90^\circ - \frac{C}{2}\right)$$

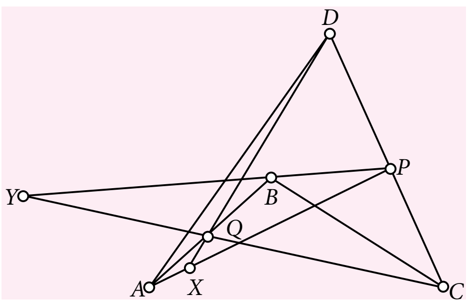
$$= 90^\circ + \frac{C}{2}$$

Also, $\angle AIG = \angle ABI + \angle BAI = \dfrac{B}{2} + \dfrac{A}{2}$

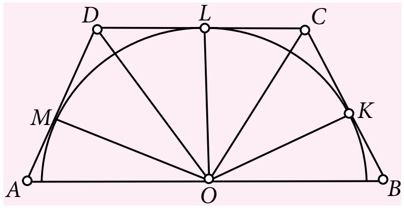
So, $\angle AEG + \angle AIG = 90^\circ + \dfrac{C}{2} + \dfrac{B}{2} + \dfrac{A}{2} = 180^\circ$

This implies that $IGEA$ is a cyclic quadrilateral. So $\angle AGI = \angle AEI = 90^\circ$.

That is AG is perpedicular to GI, as required.



8. Let K, L, M be the points of contact of the semicircle with the sides BC, CD, DA respectively. Join OK, OL, OM, OC and OD where O is the centre of the circle as well as the midpoint of AB.



In the right angled triangles AOM and BOK, we have $AO = BO$ and $OM = OK$. Hence these two triangles are congruent. Thus

$\angle AOM = \angle BOK = \alpha$, $\angle DOM = \angle DOL = \beta$ and $\angle COL = \angle COK = \gamma$.

Adding up the angles formed at O, we obtain $2\alpha + 2\beta + 2\gamma = 180^\circ$ and so $\alpha + \beta + \gamma = 90^\circ$.

In triangles AOD and BCO, we have

$\angle OAD = 90^\circ - \angle AOM = 90^\circ - \alpha$;

and $\angle CBO = 90^\circ - \angle KOB = 90^\circ - \alpha$.

Similarly $\angle AOD = \alpha + \beta = 90^\circ - \gamma$;

and $\angle BCO = 90^\circ - \angle COK = 90^\circ - \gamma$.

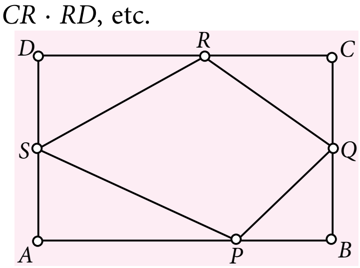
Therefore, triangle AOD is similar to triangle BCO.

Consequently, $\dfrac{AO}{AD} = \dfrac{BC}{BO}$.

So, $AD \cdot BC = AO \cdot BO = \dfrac{1}{4} AB^2$

That is $AB^2 = 4AD \cdot BC$.

9. We have (see figure) $PQ \cdot QR > BQ \cdot QC$, $QR \cdot RS > CR \cdot RD$, etc.



Therefore,

$(PQ + QR + RS + SP)^2 = PQ^2 + ... + 2PQ \cdot QR + ...$

$> (PB^2 + BQ^2) + ... + 2BQ \cdot QC + ...$

$= (PA + PB)^2 + (BQ + QC)^2 + (CR + RD)^2 + (DS + SA)^2$

$= AB^2 + BC^2 + CD^2 + DA^2$

$= AC^2 + BD^2 = 2AC^2$

Hence $PQ + QR + RS + SP > \sqrt{2}\, AC$.

10. Let us start counting 3-term G.P.'s with common ratios $2, 2^2, 2^3, ...$

The 3-term G.P.'s with common ratio 2 are

$1, 2, 2^2; 2, 2^2, 2^3; ... ; 2^{n-2}, 2^{n-1}, 2^n$.

They are $(n-1)$ in number. The 3-term GP's with common ratio 2^2 are

$$1, 2^2, 2^4; 2, 2^3, 2^5; \ldots ; 2^{n-4}, 2^{n-2}, 2^n.$$

They are $(n-3)$ in number. Similarly we see that the 3-term GP's with common ratio 2^3 are $(n-5)$ in number and so on. Thus the number of 3-term GP's which can be formed from the sequence $1, 2, 2^2, 2^3, \ldots, 2^n$ is equal to $S = (n-1) + (n-3) + (n-5) + \ldots$

Here the last term is 2 or 1 according as n is odd or even. If n is odd, then

$$S = (n-1) + (n-3) + (n-5) + \ldots + 2$$

$$= 2\left(1+2+3+\ldots+\frac{n-1}{2}\right) = \frac{n^2-1}{4}$$

If n is even, then

$$S = (n-1) + (n-3) + \ldots + 1 = \frac{n^2}{4}$$

Here the required number is $\frac{(n^2-1)}{4}$ or $\frac{n^2}{4}$ according as n is odd or even.

11. (a) Let a, b, c be the sides of a triangle with $a + b + c = 1996$, and each being a positive integer. Then $a + 1$, $b + 1$, $c + 1$ are also sides of a triangle with perimeter 1999 because

$$a < b + c \quad \Rightarrow \quad a + 1 < (b + 1) + (c + 1),$$

and so on. Moreover (999, 999, 1) form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a + 1, b + 1, c + 1)$ where a, b, c are the integers and the sides of a triangle with $a + b + c = 1996$. We conclude that $f(1999) > f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x + y + z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z \geq 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible.

We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1$, $y - 1$, $z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.

YOU ASKED WE ANSWERED

Do you have a question that you just can't get answered?

Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. If α and β be two different roots of the equation $a \cos\theta + b \sin\theta = c$, then prove that $\cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$.

(V. Rajesh, Kerala)

Ans. Here $a \cdot \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} + b \cdot \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = c$

or, $a\left(1-\tan^2\dfrac{\theta}{2}\right) + 2b\tan\dfrac{\theta}{2} = c\left(1+\tan^2\dfrac{\theta}{2}\right)$

or, $(a+c)\tan^2\dfrac{\theta}{2} - 2b\tan\dfrac{\theta}{2} + (c-a) = 0$

Roots of the equation are $\tan\dfrac{\alpha}{2}$ and $\tan\dfrac{\beta}{2}$.

$\therefore\ \tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2} = \dfrac{2b}{a+c},\ \tan\dfrac{\alpha}{2} \cdot \tan\dfrac{\beta}{2} = \dfrac{c-a}{c+a}$

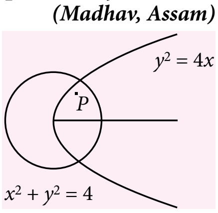
$$\therefore\ \tan\frac{\alpha+\beta}{2} = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}} = \frac{\frac{2b}{a+c}}{1-\frac{c-a}{c+a}} = \frac{b}{a}$$

$$\therefore\ \cos(\alpha+\beta) = \frac{1-\tan^2\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}} = \frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}} = \frac{a^2-b^2}{a^2+b^2}.$$

2. For what real values of a, the point $(-2a, a+1)$ will be an interior point of the smaller region bounded by the circle $x^2+y^2=4$ and the parabola $y^2=4x$?

(Madhav, Assam)

Ans. The point $P(-2a, a+1)$ will be an interior point of both the circle $x^2+y^2-4=0$ and the parabola $y^2-4x=0$.



$\therefore\quad (-2a)^2 + (a+1)^2 - 4 < 0,$

$\quad\quad i.e., 5a^2 + 2a - 3 < 0 \quad\quad ...(1)$

and $\quad (a+1)^2 - 4(-2a) < 0,$

$\quad\quad i.e., a^2 + 10a + 1 < 0 \quad\quad ...(2)$

The required values of a will satisfy both (1) and (2). From (1), $(5a-3)(a+1) < 0$

$\therefore$ by sign-scheme we get, $-1 < a < \dfrac{3}{5}$. ...(3)

Solving (2), the corresponding equation is

$a^2 + 10a + 1 = 0$

or $\quad a = \dfrac{-10 \pm \sqrt{100-4}}{2} = -5 \pm 2\sqrt{6}.$

$\therefore$ by sign-scheme for (2),

$-5-2\sqrt{6} < a < -5+2\sqrt{6}$ ...(4)

The set of values of a satisfying (3) and (4) is

$-1 < a < -5+2\sqrt{6}.$

3. Let $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ be two roots of $(x+1)^n + x^n + 1 = 0$ where α, β are roots of $x^2+px+q=0$. If α, β are also roots of the equation $x^{2n} + p^n x^n + q^n = 0$, then show that n must be an even integer when $p \neq 0$.

(Swastika, A.P)

Ans. As $\dfrac{\alpha}{\beta}$ is a root of $(x+1)^n + x^n + 1 = 0$,

$$\left(\frac{\alpha}{\beta}+1\right)^n + \left(\frac{\alpha}{\beta}\right)^n + 1 = 0$$

or, $\quad (\alpha+\beta)^n + \alpha^n + \beta^n = 0.$ ...(1)

But α, β are roots of $x^2+px+q=0$

$\therefore\quad \alpha+\beta = -p, \alpha\beta = q$

$\therefore\quad$ from (1), $(-p)^n + \alpha^n + \beta^n = 0$ ...(2)

A α, β are roots of $x^{2n} + p^n x^n + q^n = 0$, we get

$\alpha^{2n} + p^n\alpha^n + q^n = 0$

and $\beta^{2n} + p^n\beta^n + q^n = 0.$

Subtracting, $\alpha^{2n} - \beta^{2n} + p^n(\alpha^n - \beta^n) = 0$

or $\quad (\alpha^n+\beta^n)(\alpha^n-\beta^n) + p^n(\alpha^n-\beta^n) = 0$

$\therefore\quad \alpha^n + \beta^n + p^n = 0$ ...(3)

$[\because \alpha^n \neq \beta^n \text{ as } \alpha \neq \beta]$

From $(2)-(3), (-p)^n - p^n = 0$

$\therefore\quad (-p)^n = p^n \ \Rightarrow\ n$ is an even integer.

Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, then $\frac{dy}{dx}=$

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$

(c) $\frac{1}{1+x}$ (d) none of these

2. If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}=$

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\sin(a+y)$

(c) $\sin^2(a+y)$ (d) $\frac{\sin(a+y)}{\sin a}$

3. $\lim_{x\to\infty}\sqrt{\left(\frac{x+\sin x}{x-\cos x}\right)}=$

(a) 0 (b) 1 (c) -1 (d) 2

4. If $f(x)=\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$, then $\lim_{x\to\infty} f(x)$ is

(a) e^4 (b) e^3 (c) e^2 (d) 2^4

5. Let α and β be the roots of $ax^2+bx+c=0$, then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

(a) 0 (b) $\frac{1}{2}(\alpha-\beta)^2$

(c) $\frac{a^2}{2}(\alpha-\beta)^2$ (d) $-\frac{a^2}{2}(\alpha-\beta)^2$

6. Find the domain of the function $f(x)=\frac{2}{\left[\frac{x}{3}\right]}-5^{\sin^{-1}x^2}+\frac{(3x+1)!}{\sqrt{x+1}}$, where [.] denotes the greatest integer function.

7. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of equation $x^n - nax - b = 0$ and $(\alpha_1-\alpha_2)\cdot(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)=A$, then find the value of $A - n\alpha_1^{\,n-1}$.

8. If left hand derivative and right hand derivative of a function f at 'a' are finite, then show that f is continuous at 'a'.

9. Find the value(s) of 'a' for which $\lim_{x\to 0}\frac{\sin 3x + a\sin 2x}{x^3}$ exists finitely. Find the value of the limit also.

10. If α, β are distinct real roots of the quadratic equation $ax^2+bx+c=0$, then show that

$$\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}=\frac{b^2-4ac}{2}.$$

SOLUTIONS

1. **(d) :** Squaring the given equation we will get $y=\frac{-x}{1+x}$. Hence, $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$

2. **(a)**

3. **(b) :** $\lim_{x\to\infty}\sqrt{\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}}=1$

$\because \lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\frac{\cos x}{x}=0$

4. **(a) :** Use the fact $\lim_{x\to\infty}(f(x))^{g(x)} = e^{\lim_{x\to\infty}(f(x)-1)g(x)}$

Where $f(x)\to 1$, $g(x)\to\infty$ as $x\to\infty$

5. **(c) :** Use the fact $\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\frac{a^2}{2}$

6. As $\left[\frac{x}{3}\right]=0 \Rightarrow 0\le\frac{x}{3}<1\Rightarrow 0\le x<3$

i.e., $\left[\frac{x}{3}\right]$ is defined for $x\in R-[0, 3)$.

$\sin^{-1}x^2$ is defined for $0\le x^2\le 1 \Rightarrow x\in[-1, 1]$

$(3x+1)!$ is defined for $3x+1\ge 0$ and

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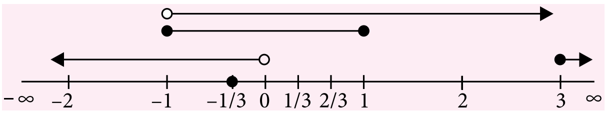
$(3x + 1) \in \{0, 1, 2, 3, \ldots\}$

$$\Rightarrow \quad x \in \left\{\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots\ldots\right\}$$

$\dfrac{1}{\sqrt{x+1}}$ is defined for $x \in (-1, \infty)$



$\therefore$ $f(x)$ is defined for $x \in \left\{\dfrac{-1}{3}\right\}$

7. Given $x^n - nax - b = 0$. Roots $\rightarrow \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$

$\therefore$ $x^n - nax - b = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$

or $\dfrac{x^n - nax - b}{(x-\alpha_1)} = (x-\alpha_2)(x-\alpha_3)\ldots.(x-\alpha_n)$

or $\displaystyle\lim_{x\to\alpha_1} \frac{x^n - nax - b}{x-\alpha_1} = \lim_{x\to\alpha_1} (x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

or $\displaystyle\lim_{x\to\alpha_1} \frac{nx^{n-1} - na}{1} = \lim_{x\to\alpha_1} (x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

or $n\,\alpha_1^{n-1} - na = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) \ldots (\alpha_1 - \alpha_n)$

$\Rightarrow$ $n\,\alpha_1^{n-1} - na = A$ $\quad\therefore$ $A - n\alpha_1^{n-1} = -na$

8. $\displaystyle\lim_{h\to 0^+} (f(a) - f(a-h)) = \lim_{h\to 0^+} \frac{f(a)-f(a-h)}{h}\,.\,h$

$\displaystyle = \lim_{h\to 0^+} f'(a^-)\cdot h$ (as $f'(a^-)$ is finite) $= 0$

Similarly $\displaystyle\lim_{h\to 0^+} (f(a+h) - f(a)) = \lim_{h\to 0^+} f'(a^+)\cdot h = 0$

Hence $\displaystyle\lim_{h\to 0^+} f(a-h) = \lim_{h\to 0^+} f(a+h) = f(a)$

Hence f is continuous at a.

9. Let $\displaystyle l = \lim_{x\to 0} \frac{\sin 3x + a\sin 2x}{x^3} \quad \left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0} \frac{3\cos 3x + 2a\cos 2x}{3x^2}$$

we should have $g(0) = 0$, where

$g(x) = 3\cos 3x + 2a \cos 2x$,

as the given limit exists finitely $\Rightarrow$ $a = -3/2$.

Further, $\displaystyle l = \lim_{x\to 0} \frac{3\cos 3x - 3\cos 2x}{3x^2}$

$$= \lim_{x\to 0} \frac{\cos 3x - \cos 2x}{x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\sin 2x - 3\sin 3x}{2x} \quad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{4\cos 2x - 9\cos 3x}{2} = -\frac{5}{2}$$

Thus $a = -3/2$ and limit value $= -5/2$.

10. $\displaystyle\lim_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} = \lim_{x\to\alpha} \frac{2\sin^2 \frac{ax^2+bx+c}{2}}{(x-\alpha)^2}$

$$= \lim_{x\to\alpha} \frac{2\sin^2 \frac{a(x-\alpha)(x-\beta)}{2}}{\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)^2} \cdot \frac{a^2}{4}(x-\beta)^2$$

$$= \frac{a^2(\alpha-\beta)^2}{2} = \frac{a^2}{2}[(\alpha+\beta)^2 - 4\alpha\beta]$$

$$= \frac{a^2}{2}\left[\frac{b^2}{a^2} - \frac{4c}{a}\right] = \frac{b^2 - 4ac}{2}$$

MATHS MUSING

SOLUTION SET-148

1. **(c) :** $\cos A \cos^2 C = -\frac{1}{2} \Rightarrow \cos 2C\,(1 + \cos 2C) - 1 = 0$

$\therefore\ \cos 2C = \frac{\sqrt{5}-1}{2} \Rightarrow C = \frac{1}{2}\cos^{-1}(2\sin 18°)$

2. **(c) :** $[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} 1 & \cos\beta & \cos\beta \\ \cos\beta & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$

$= (1 - \cos\alpha)(\cos\alpha - \cos 2\beta) \geq 0$

$\therefore\ \alpha \leq 2\beta$

3. **(d):** $N = -3(1-10)^{1007}$

$= -3\left[1 - 1007\cdot 10 + \binom{1007}{2}10^2 - \binom{1007}{3}10^3\right]$

$= -3(-2969) = 8907$ with digit sum 24.

4. **(d):** $a + ib = re^{i\theta} \Rightarrow (re^{i\theta})^{c+id} = p$

$(c + id)(\ln r + i\theta) = \ln p \Rightarrow c\ln r - d\theta = \ln p$

and $d\ln r + c\theta = 0$. Eliminating θ, we get

$(c^2 + d^2)\ln r = c\ln p,\ r^{c^2+d^2} = p^c$

$(r^2)^{(c^2+d^2)} = p^{2c},\ (a^2+b^2)^{(c^2+d^2)} = p^{2c}$

5. **(a), (b), (d) :** $6^2 + 5^2 + 4^2 - 3^2 - 2^2 - 1^2 = 3$ $(1+2+3+4+5+6)$

$\therefore\ S = 3\,(1 + 2 + 3 + \ldots + 2016) = 3\cdot 1008\cdot 2017$
$= 2^4\cdot 3^3\cdot 7\cdot 2017$

6. **(a), (d) :** Let $a = \frac{1}{2\sqrt{2}}$, $x^2 = 4ay$, $x = 2at$,

$y = at^2$. Normal at t is $y = -\frac{x}{t} + \frac{t^2+2}{2\sqrt{2}}$.

This is same as $y = mx + \frac{1}{m}$, tangent to

$y^2 = 4x$. $\therefore\ t = -\frac{1}{m} = -\sqrt{2}$

$\therefore$ The line is $y = \frac{x}{\sqrt{2}} + \sqrt{2}$ or $x - \sqrt{2}\,y + 2 = 0$

Distance of (0, 0) from it is $\frac{2}{\sqrt{3}}$

Also $x = 0$ is a line giving the distance 0.

7. **(d):** $G_n = ((n+1)(n+2)\ldots(n+n))^{\frac{1}{n}}$

$$\frac{G_n}{n} = \left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty}\ln\frac{G_n}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right) = \int_0^1 \ln(1+x)\,dx = \ln\frac{4}{e}$$

$\therefore\ \lim_{n\to\infty}\frac{G_n}{n} = \frac{4}{e}$

8. **(b):** $H_n = \dfrac{n}{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}}$

$$\lim_{n\to\infty}\frac{n}{H_n} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$$

$$= \int_0^1 \frac{dx}{1+x} = \ln 2\,.$$

$\therefore\ \lim_{n\to\infty}\frac{H_n}{n} = \frac{1}{\ln 2}$

9. **(3):** $(10^{2015} + 5)^2 = 225\,N$

$\therefore\ 9N = (2\cdot 10^{2014} + 1)^2 = 4\cdot 10^{4028} + 4\cdot 10^{2014} + 1$

$\therefore\ N = \frac{4(10^{4028}-1)}{10-1} + \frac{4(10^{2014}-1)}{10-1} + 1$

$= \underbrace{44\,..\,4}_{2014}\underbrace{88\,..\,8}_{2013}9 = 4\times 2014 + 8\times 2013 + 9$

$= 24169$, with 3 even digits 2, 4, 6

10. **(c) :** P. $x - \frac{1}{x} = i$

$\Rightarrow x = -i\omega,\ -i\omega^2,\ x^{2015} - \frac{1}{x^{2015}} = -i$

Q. $x - \frac{1}{x} = -i \Rightarrow x = i\omega,\ i\omega^2,\ x^{2015} - \frac{1}{x^{2015}} = i$

R. $x = -i\omega,\ -i\omega^2 \Rightarrow x^{2016} - \frac{1}{x^{2016}} = 0$

S. $x = i\omega,\ i\omega^2 \Rightarrow x^{2014} + \frac{1}{x^{2014}} = 1$

Solution Sender of Maths Musing

SET-147

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Contd. from page no. 30

CLASS XII

SOLVED PAPER

CBSE BOARD 2015

$$\left|\frac{\sqrt{3}x^2}{2}\right|_0^1 + \frac{1}{\sqrt{3}}\left|4x - \frac{x^2}{2}\right|_1^4$$

$$= \frac{\sqrt{3}}{2}\times 1 + \frac{4}{\sqrt{3}}(4-1) - \frac{1}{\sqrt{3}\times 2}(4^2-1^2)$$

$$= \frac{\sqrt{3}}{2} + \frac{12}{\sqrt{3}} - \frac{1}{2\sqrt{3}}\times 15$$

$$= \frac{3+24-15}{2\sqrt{3}} = \frac{12}{2\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

OR

Let $I = \int_1^3 (e^{2-3x} + x^2 + 1)\,dx$

Since $\int_a^b f(x)\,dx = (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\times$

$[f(a) + f(a+h) + ... + f(a+(n-1)h)]$

where, $h = \frac{b-a}{n}$

here, $a = 1$, $b = 3$, $b - a = 2$

$\Rightarrow \quad h = \frac{2}{n}$ ...(i)

$f(a) = f(1) = e^{2-3} + 1^2 + 1$

$\therefore \quad I = (3-1)\lim\limits_{n\to\infty}\frac{1}{n}[e^{2-3} + 1^2 + 1 + e^{2-3(1+h)} + (1+h)^2$
$+ 1 + ... + e^{2-3(1+(n-1)h)} + (1+(n-1)h)^2 + 1]$

$= 2\lim\limits_{n\to\infty}\frac{1}{n}[e^2e^{-3} + e^2e^{-3(1+h)} + e^2e^{-3(1+2h)} + ... +$
$e^2e^{-3(1+(n-1)h)} + 1^2 + (1+h)^2 + (1+2h)^2 + ...$
$+ (1+(n-1)h)^2 + n]$

$= 2\lim\limits_{n\to\infty}\frac{1}{n}\times$

$[e^2e^{-3}(1 + e^{-3h} + e^{-6h} + ... + e^{-3(n-1)h}) + n + h^2 \times$
$[1^2 + 2^2 + ... + (n-1)^2] + 2h[1 + 2 + 3 + ... + n] + n]$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{-1}(1-(e^{-3h})^n)}{1-e^{-3h}} + h^2\times\frac{(n-1)n(2n-1)}{6} + 2h\times\frac{n(n+1)}{2} + 2n\right]$$

$$= 2\lim_{n\to\infty}\left[\frac{e^{-1}}{n}\frac{(1-(e^{-3hn}))}{1-e^{-3h}} + \frac{h^2n^2}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + hn\left(1+\frac{1}{n}\right) + 2\right]$$

$$= 2\left[\lim_{n\to\infty}\frac{e^{-1}}{n}\frac{\left(1-e^{-3\times\frac{2}{n}\times n}\right)}{\left(1-e^{-3\times\frac{2}{n}}\right)} + \frac{2^2}{n^2}\times\frac{n^2\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6} + \frac{2}{n}\times n\left(1+\frac{1}{n}\right) + 2\right]$$

(Using (i))

$$= 2\left[\lim_{n\to\infty}\frac{e^{-1}(1-e^{-6})}{\left(\frac{1-e^{-6/n}}{1/n}\right)} + \frac{4\times 2}{6} + 2 + 2\right]$$

$$= 2\left[\frac{(e^{-1}-e^{-7})}{\lim\limits_{h\to 0}\left(\frac{1-e^{-3h}}{h/2}\right)} + \frac{4}{3} + 4\right]$$

$$= 2\left[\frac{e^{-1}-e^{-7}}{\lim\limits_{h\to 0}\left(\frac{1-e^{-3h}}{-3h}\right)\times(-3\times 2)} + \frac{4}{3} + 4\right]$$

$$= \frac{e^{-1}-e^{-7}}{\lim\limits_{h\to 0}\left(\frac{1-\frac{1}{e^{3h}}}{\frac{h}{2}}\right)} + \frac{32}{3}$$

$$= \frac{e^{-1}-e^{-7}}{\lim\limits_{h\to 0}\left(\frac{e^{3h}-1}{3h}\right)\times\frac{3h}{e^{3h}}\times\frac{2}{h}} + \frac{32}{3}$$

$$= \frac{e^{-1}-e^{-7}}{6} + \frac{32}{3}$$

JEE MAIN

SOLVED PAPER 2015

* ALOK KUMAR, B.Tech, IIT Kanpur

1. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is

(a) $\frac{2}{3}$ (b) $\frac{-2\sqrt{3}}{3}$ (c) $\frac{2\sqrt{2}}{3}$ (d) $\frac{-\sqrt{2}}{3}$

2. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is

(a) $y^2 = 2x$ (b) $x^2 = 2y$
(c) $x^2 = y$ (d) $y^2 = x$

3. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower are 30°, 45° and 60° respectively, then the ratio, $AB : BC$, is

(a) $1 : \sqrt{3}$ (b) $2 : 3$ (c) $\sqrt{3} : 1$ (d) $\sqrt{3} : \sqrt{2}$

4. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0), is

(a) 820 (b) 780 (c) 901 (d) 861

5. The equation of the plane containing the line $2x - 5y + z = 3$; $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$, is

(a) $x + 3y + 6z = 7$ (b) $2x + 6y + 12z = -13$
(c) $2x + 6y + 12z = 13$ (d) $x + 3y + 6z = -7$

6. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is

(a) 275 (b) 510 (c) 219 (d) 256

7. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a

(a) circle of radius $\sqrt{2}$.
(b) circle of radius $\sqrt{3}$.
(c) straight line parallel to x-axis.
(d) straight line parallel to y-axis.

8. $\lim_{x \to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to

(a) 2 (b) $\frac{1}{2}$ (c) 4 (d) 3

9. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$, is

(a) $3\sqrt{21}$ (b) 13
(c) $2\sqrt{14}$ (d) 8

10. The sum of coefficients of integral powers of x in the binomial expansion of $\left(1 - 2\sqrt{x}\right)^{50}$ is

(a) $\frac{1}{2}\left(3^{50} - 1\right)$ (b) $\frac{1}{2}\left(2^{50} + 1\right)$
(c) $\frac{1}{2}\left(3^{50} + 1\right)$ (d) $\frac{1}{2}\left(3^{50}\right)$

11. The sum of first 9 terms of the series

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots \text{ is}$$

(a) 142 (b) 192 (c) 71 (d) 96

12. The area (in sq. units) of the region described by $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is

(a) $\frac{15}{64}$ (b) $\frac{9}{32}$ (c) $\frac{7}{32}$ (d) $\frac{5}{64}$

---

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13. The set of all values of λ for which the system of linear equations

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

has a non-trivial solution,
(a) contains two elements.
(b) contains more than two elements.
(c) is an empty set.
(d) is a singleton.

14. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a
(a) circle of radius 2.
(b) circle of radius $\sqrt{2}$.
(c) straight line parallel to x-axis.
(d) straight line parallel to y-axis.

15. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is
(a) 3 (b) 4 (c) 1 (d) 2

16. The number of integers greater than 6,000 that can be formed, using the digits 3, 5, 6, 7 and 8 without repetition, is
(a) 120 (b) 72 (c) 216 (d) 192

17. Let $y(x)$ be the solution of the differential equation

$$(x \log x)\frac{dy}{dx} + y = 2x \log x, \ (x \geq 1)$$

Then $y(e)$ is equal to
(a) 2 (b) $2e$ (c) e (d) 0

18. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is a 3×3 identity matrix, then the ordered pair (a, b) is equal to
(a) $(2, 1)$ (b) $(-2, -1)$ (c) $(2, -1)$ (d) $(-2, 1)$

19. If m is A. M. of two distinct real numbers l and n $(l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals
(a) $4\,lmn^2$ (b) $4l^2m^2n^2$ (c) $4\,l^2mn$ (d) $4\,lm^2n$

20. The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to
(a) $s \vee (r \vee \sim s)$ (b) $s \wedge r$
(c) $s \wedge \sim r$ (d) $s \wedge (r \wedge \sim s)$

21. The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals
(a) $-(x^4+1)^{\frac{1}{4}} + c$ (b) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$
(c) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$ (d) $(x^4+1)^{\frac{1}{4}} + c$

22. The normal to the curve, $x^2 + 2xy - 3y^2$ at $(1, 1)$
(a) meets the curve again in the first quadrant.
(b) meets the curve again in the fourth quadrant.
(c) does not meet the curve again.
(d) meets the curve again in the second quadrant.

23. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is
(a) $\frac{3x - x^3}{1 + 3x^2}$ (b) $\frac{3x + x^3}{1 + 3x^2}$
(c) $\frac{3x - x^3}{1 - 3x^2}$ (d) $\frac{3x + x^3}{1 - 3x^2}$

24. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is
(a) $\frac{10}{3}$ (b) 4 (c) 2 (d) $\frac{16}{5}$

25. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is
(a) 15.8 (b) 14.0
(c) 16.8 (d) 16.0

26. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to
(a) 1 (b) 6 (c) 2 (d) 4

27. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to
(a) 3 (b) -3 (c) 6 (d) -6

28. Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$ then $f(2)$ is equal to
(a) 0 (b) 4 (c) -8 (d) -4

29. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is

(a) $\frac{27}{2}$ (b) 27 (c) $\frac{27}{4}$ (d) 18

30. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls is

(a) $220\left(\frac{1}{3}\right)^{12}$ (b) $22\left(\frac{1}{3}\right)^{11}$

(c) $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$ (d) $55\left(\frac{2}{3}\right)^{10}$

SOLUTIONS

1. (c) : Expanding $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

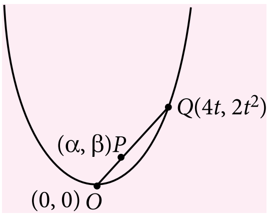
$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-\left\{(\vec{b}\cdot\vec{c})+\frac{1}{3}|\vec{b}||\vec{c}|\right\}\vec{a}=0$

As $\vec{a}$ and $\vec{b}$ are non-collinear, the coefficients must vanish. Thus, $\vec{a}\cdot\vec{c}=0$ and $(\vec{b}\cdot\vec{c})=-\frac{1}{3}|\vec{b}||\vec{c}|$

Again, $\cos\theta=\frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} \Rightarrow \cos\theta=-\frac{1}{3}$

$\Rightarrow \sin\theta=\frac{2\sqrt{2}}{3}$ **[Rating : Medium]**

2. (b) :



$OP : PQ = 1 : 3$

Let the parametric co-ordinates of Q be $(4t, 2t^2)$

We have, by section formula

$$\alpha=\frac{4t+0}{4}=t \text{ and } \beta=\frac{2t^2+0}{4}=\frac{t^2}{2}$$

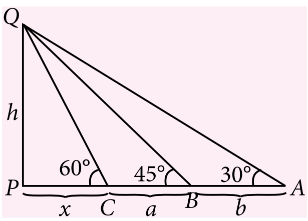
Eliminating 't', we get the locus of $P(\alpha, \beta)$ as

$\alpha^2 = 2\beta$

Thus the locus is $x^2 = 2y$

[Rating : Easy]

3. (c) :



Using basic trigonometry in appropriate triangles

$\tan 60°=\frac{h}{x} \Rightarrow x=\frac{h}{\sqrt{3}}$

$\tan 45°=\frac{h}{x+a} \Rightarrow x+a=h$

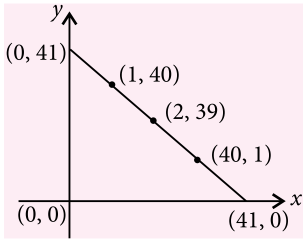
and $\tan 30°=\frac{h}{x+a+b} \Rightarrow x+a+b=h\sqrt{3}$

We have, $a=h\left(1-\frac{1}{\sqrt{3}}\right)=\frac{h(\sqrt{3}-1)}{\sqrt{3}}$

$b=h\sqrt{3}-h=h(\sqrt{3}-1)$

$\therefore \frac{AB}{BC}=\frac{b}{a}=\sqrt{3} \therefore AB:BC=\sqrt{3}:1$ **[Rating : Easy]**

4. (b) : 1st solution :



We count the number of points on line $x = n$, $1 \le n < 40$, that lie in the interior of the triangle.

At line $x = 40$, we have 1 point.

$x = 39$, we have 2 points

......

$x = n$, we have $(40 - n)$ points

The total number of points $= 1 + 2 + 3 + ... + 39$

$$=\frac{1}{2}\times 39\times 40=39\times 20=780$$

2nd solution : The points $P(\alpha, \beta)$ satisfying the requirements of the problem are given by

$\alpha \ge 1$

$\beta \ge 1$

$\alpha + \beta < 41$ *i.e.* $\alpha + \beta \le 40$

The number of integral solution to the equation

$\alpha + \beta + \gamma = 40$, ($\gamma \ge 0$ is a slack variable)

is the answer to the problem. Hence the number of points is equal to the non-negative integral solution of $x + y + z = 40$, $x, y, z \ge 0$

which is given by ${}^{38+3-1}C_{3-1} = {}^{40}C_2 = 780$

[Rating : Medium]

5. (a) : 1st solution : Let the equation of line parallel to the plane

$x + 3y + 6z = 1$ be $x + 3y + 6z = k$

As a point on line of intersection of planes

$2x - 5y + z = 3$ and $x + y + 4z = 5$ is $(4, 1, 0)$

got by inspection, we have the required plane satisfying this point.

Hence $k = 4 + 3\cdot 1 + 0 = 7$

Thus the equation of plane is $x + 3y + 6z = 7$

2nd solution : The equation of plane containing the line $2x - 5y + z = 3$, $x + y + 4z = 5$ is

$$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$$

$\Rightarrow \quad (2 + \lambda)x + (\lambda - 5)y + (4\lambda + 1)z - (5\lambda + 3) = 0$

As this plane is parallel to $x + 3y + 6z - 1 = 0$, the coefficients must be proportional, giving

$$\frac{2+\lambda}{1} = \frac{\lambda-5}{3} = \frac{4\lambda+1}{6} = \frac{5\lambda+3}{1}$$

Taking any two of them give, (for example 1st and 2nd)

$$6 + 3\lambda = \lambda - 5$$

$\Rightarrow \quad 2\lambda = -11 \quad \Rightarrow \quad \lambda = -\frac{11}{2}$

The equation of plane is $-\frac{7x}{1} - \frac{21y}{2} - 21z + \frac{49}{2} = 0$

i.e., $7x + 21y + 42z - 49 = 0$

i.e., $x + 3y + 6z = 7$ **[Rating : Medium]**

6. (c) : We have, $n(A) = 4$ and $n(B) = 2$

Thus the number of elements in $A \times B = 8$

Number of subsets having at least 3 elements

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$

$= ({}^8C_0 + {}^8C_1 + {}^8C_2 + \ldots.. + {}^8C_8) - ({}^8C_0 + {}^8C_1 + {}^8C_2)$

$= 2^8 - (1 + 8 + 28) = 256 - 37 = 219$

[Rating : Medium]

7. (a) : The line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$ passes through the intersection of line

$$\left.\begin{aligned} 2x - 3y + 4 &= 0 \\ \text{and } x - 2y + 3 &= 0 \end{aligned}\right\} \text{(A)}$$

for different values of k.

Lines given by (A) meet at $(1, 2)$

Let image of $A(2, 3)$ in the family of lines be $B(\alpha, \beta)$.

Thus $(1, 2)$ is a fixed point for given family of lines, we have $AP = BP \quad \Rightarrow \quad (\alpha - 1)^2 + (\beta - 1)^2 = 2$

The locus generalises to $(x - 1)^2 + (y - 1)^2 = 2$

Thus it is a circle of radius $\sqrt{2}$. **[Rating : Medium]**

8. (a) : $\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$

$$= \lim_{x\to 0} \frac{2(\sin^2 x)(3+\cos x)}{x \tan 4x}$$

$$= \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)^2 \frac{(3+\cos x)}{\left(\frac{\tan 4x}{x}\right)} = 2 \cdot 1 \cdot \frac{(3+1)}{4} = 2$$

[Rating : Easy]

9. (b) : Let the parameter corresponding to the point of intersection be denoted by t, then

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = t$$

Thus $(3t + 2, 4t - 1, 12t + 2)$ is a general point.

Thus point lies on plane $x - y + z = 16$ gives

$$(3t + 2) - (4t - 1) + (12t + 2) = 16$$

$\Rightarrow \quad 11t = 11 \quad \therefore \quad t = 1$

Thus the point is $(5, 3, 14)$

Given point is $(1, 0, 2)$

The distance between the points

$$\sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2}$$

$= \sqrt{16 + 9 + 144} = \sqrt{169} = 13$ **[Rating : Medium]**

10. (c) : By Binomial theorem

$$(1-2\sqrt{x})^{50} = {}^{50}C_0 - {}^{50}C_1(2\sqrt{x}) + {}^{50}C_2(2\sqrt{x})^2 + \ldots\ldots\ldots. + {}^{50}C_{50}(2\sqrt{x})^{50}$$

$$(1+2\sqrt{x})^{50} = {}^{50}C_0 + {}^{50}C_1(2\sqrt{x}) + {}^{50}C_2(2\sqrt{x})^2 + \ldots\ldots\ldots. + {}^{50}C_{50}(2\sqrt{x})^{50}$$

On addition

$$(1+2\sqrt{x})^{50} + (1-2\sqrt{x})^{50} = 2({}^{50}C_0 + {}^{50}C_2(2\sqrt{x})^2 + {}^{50}C_3(2\sqrt{x})^3 + \ldots\ldots + {}^{50}C_{50}(2\sqrt{x})^{50})$$

Set $x = 1$ to obtain

$3^{50} + 1 = 2$ (sum of coefficients of integral powers of x)

$\therefore$ Sum of coeff. of integral powers of $x = \frac{1}{2}(3^{50} + 1)$

[Rating : Medium]

11. (d) : The n^{th} term, t_n is

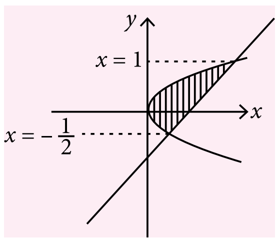
$$= \frac{1^3 + 2^3 + \ldots. + n^3}{1 + 3 + \ldots.. + (2n-1)} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4}$$

$$\sum_{n=1}^{9} t_n = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}\left\{\sum_{n=1}^{10} n^2 - 1\right\}$$

$$= \frac{1}{4}\left\{\frac{10 \cdot 11 \cdot 21}{6} - 1\right\} = \frac{1}{4}\{385 - 1\} = \frac{1}{4} \times 384 = 96$$

[Rating : Medium]

12. (b) :



The area is given by

$$\int (x_1 - x_2)dy \text{ and in this case}$$

$$\int_{-1/2}^{1} \left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy = \left.\frac{(y+1)^2}{8} - \frac{y^3}{6}\right|_{-1/2}^{1}$$

$$=\left[\frac{2^2}{8}-\frac{1}{6}\right]-\left[\frac{\frac{1}{4}}{8}+\frac{1}{8\cdot 6}\right]=\left(\frac{1}{2}-\frac{1}{6}\right)-\left(\frac{1}{32}+\frac{1}{48}\right)$$

$$=\frac{3-1}{6}-\left(\frac{3+2}{96}\right)=\frac{1}{3}-\frac{5}{96}=\frac{32-5}{96}=\frac{27}{96}=\frac{9}{32}$$

[Rating : Easy]

13. (a) : The system is $(2-\lambda)x_1 - 2x_2 + x_3 = 0$

$2x_1 - (3+\lambda)x_2 + 2x_3 = 0$

$-x_1 + 2x_2 - \lambda x_3 = 0$

For non-trivial solution, the determinant of the coefficient matrix must vanish. Then

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix}=0$$

$\Rightarrow$ $(2-\lambda)\{\lambda(3+\lambda)-4\} + 2\{-2\lambda-3\}$
$\qquad + 1\{4-(3+\lambda)\} = 0$

$\Rightarrow$ $(2-\lambda)(\lambda^2+3\lambda-4) - 4\lambda - 6 + 1 - \lambda = 0$

$\Rightarrow$ $(2-\lambda)(\lambda^2+3\lambda-4) - 5\lambda - 5 = 0$

$\Rightarrow$ $(2-\lambda)(\lambda-1)(\lambda+4) - 5(\lambda+1) = 0$

$\Rightarrow$ $(\lambda-1)(-\lambda^2 - 2\lambda + 8 - 5) = 0$

$\Rightarrow$ $(\lambda-1)(\lambda^2+2\lambda-3) = 0$

$\Rightarrow$ $(\lambda-1)^2(\lambda+3) = 0$

Thus, $\lambda = 1, 1, -3$

$\therefore$ Set of all λ's contain 2 elements.

[Rating : Difficult]

14. (a) : 1st solution : We have, $\left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$

$\Rightarrow$ $|z_1-2z_2|^2 = |2-z_1\bar{z}_2|^2$

$\Rightarrow$ $(z_1-2z_2)(\bar{z}_1-2\bar{z}_2) = (2-z_1\bar{z}_2)(2-\bar{z}_1z_2)$

$\Rightarrow$ $|z_1|^2 - 4 - |z_1|^2\,|z_2|^2 + 4|z_2|^2 = 0$

$\Rightarrow$ $\{|z_1|^2-4\} - |z_2|^2\{|z_1|^2-4\} = 0$

$\Rightarrow$ $(1-|z_2|^2)(|z_1|^2-4) = 0$

Thus $|z_1| = 2$ as $|z_2| \neq 1$ (given)

The point z lies on circle of radius 2.

2nd solution :

Observe that if $\dfrac{|\alpha-\beta|}{|1-\alpha\bar{\beta}|}=1$, two complex numbers α and β of which $|\beta|\neq 1$, then $|\alpha| = 1$

Since $|\alpha-\beta| = |1-\alpha\bar{\beta}|$

$\Rightarrow$ $|\alpha-\beta|^2 = |1-\alpha\bar{\beta}|^2$

$\Rightarrow$ $|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta}) = 1 + |\alpha|^2|\beta|^2 - 2\,\mathrm{Re}(\alpha\bar{\beta})$

$\Rightarrow$ $1 - |\alpha|^2 - |\beta|^2 - |\alpha|^2|\beta|^2 = 0$

$\Rightarrow$ $(1-|\alpha|^2)(1-|\beta|^2) = 0$

As $|\beta| \neq 1$ $\therefore$ $|\alpha| = 1$

In our case take $\alpha = z_1/2$ and $\beta = z_2$

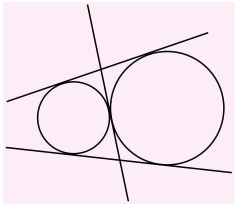
gives $|z_1/2| = 1$ $\therefore$ $|z_1| = 2$ **[Rating : Difficult]**

15. (a) : The circles can be written as $(x-2)^2 + (y-3)^2 = 25$ with centre $O_1(2, 3)$ and $r_1 = 5$ and $(x+3)^2 + (y+9)^2 = 64$ with centre $O_2(-3, -9)$ and $r_2 = 8$

O_1O_2 = distance between centres $=\sqrt{5^2+12^2}=13$

As $r_1 + r_2 = O_1O_2$

We have that circle touch each other externally, so there are three common tangents as shown.



[Rating : Medium]

16. (d) : Numbers having 5 digits = $\underline{\lfloor 5}$ = 120

Numbers having 4 digits = (3)(4)(3)(2) = 72

As the first digit can be filled in 3 ways, viz 6, 7, and 8, and as repetition is not allowed, the other choices are 4, 3 and 2 in that order.

[Rating : Difficult]

17. (a) : The equation can be written as

$$\frac{dy}{dx}+\left(\frac{1}{x\ln x}\right)y=2$$

It is linear in y. Thus

$$\text{I.F.} = e^{\int\frac{dx}{x\ln x}} = e^{\ln(\ln x)} = \ln x$$

The solution is

$$y\cdot\text{I.F.} = \int 2\cdot\text{I.F.}\,dx + \lambda$$

i.e., $\quad y(\ln x) = \int 2(\ln x) + \lambda$

$\qquad = 2x(\ln x - 1) + \lambda$

At $x = 1$, we have $\lambda = 2$

The solution become $y\ln x = 2x(\ln x - 1) + 2$

Set $x = e$ in the above to obtain

$y = 2e(\ln e - 1) + 2 = 2$

The value of y at $x = e$, *i.e.* $y(e) = 2$ **[Rating : Difficult]**

18. (b) : As $AA^T = 9I$, we have

$$\left(\frac{A}{3}\right)\left(\frac{A}{3}\right)^T = I$$

Hence, $\frac{1}{3}A$ is an orthogonal matrix.

$$\text{Now } \frac{1}{3}A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{a}{3} & \frac{2}{3} & \frac{b}{3} \end{bmatrix}$$

We know that row (column) form mutually orthogonal unit vectors. Then $\left(\frac{a}{3}, \frac{2}{3}, \frac{b}{3}\right)$ is a unit vector,
gives $a^2 + 4 + b^2 = 9$
Also, $a + 2b + 4 = 0$ and $2a - 2b + 2 = 0$
The solution is $(-2, -1)$, which is consistent with all the equations. **[Rating : Difficult]**

19. (d) : Given that l, G_1, G_2, G_3, n are in G.P.
$G_1 = lr, G_2 = lr^2, G_3 = lr^3, n = lr^4$
Then $G_1^4 + 2G_2^4 + G_3^4 = (lr)^4 + 2(lr^2)^4 + (lr^3)^4$
$= (l^3)(lr^4) + 2l^2(lr^4)^2 + l\cdot(lr^4)^3$
$= l^3\cdot n + 2l^2\cdot n^2 + ln^3 = ln(l^2 + 2nl + n^2) = ln(n + l)^2$
$= 4m^2nl$ **[Rating : Difficult]**

20. (b) : Using the rules of logic, we have
$\sim s \vee (\sim r \wedge s)$
$= (\sim s \vee \sim r) \wedge (\sim s \wedge s) = (\sim s \vee \sim r) \wedge t = \sim s \vee \sim r$
Now the negation of above is $\sim (\sim s \vee \sim r) = s \wedge r$
[Rating : Easy]

21. (b) : $I = \int \frac{dx}{x^2(x^4+1)^{3/4}}$

$$= \int \frac{dx}{x^2(x^4)^{3/4}\left\{1+\frac{1}{x^4}\right\}^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Set $1+\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt$

$$= -\frac{1}{4}\int \frac{1}{t^{3/4}}dt = -\frac{1}{4}\cdot\frac{t^{1/4}}{1/4} = -t^{1/4} = -\left(1+\frac{1}{x^4}\right)^{1/4} + \lambda$$

[Rating : Medium]

22. (b) : The given curve is
$x^2 + 2xy - 3y^2 = 0$
Factorizing it becomes $(x - y)(x + 3y) = 0$
Normal at $(1, 1)$ is $x + y = \lambda$ *i.e.* $1 + 1 = \lambda \quad \therefore \lambda = 2$
Thus the equation is $x + y = 2$
Obviously $x + 3y = 0$ doesn't have the point $(1, 1)$ on it.
Now, $x + y = 2$ meets $x + 3y = 0$ in the point $(3, -1)$ obtained by solving the system of linear equations. Hence the point is in 4th quadrant.
[Rating : Difficult]

23. (c) : As $|x| < \frac{1}{\sqrt{3}}$, in this range using principal branch of tangent function, we have

$$3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Also, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x$

Thus, $\tan^{-1}y = \tan^{-1}x + 2\tan^{-1}x = 3\tan^{-1}x$

$$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$\therefore \; y = \frac{3x-x^3}{1-3x^2}$ **[Rating : Medium]**

24. (c) : Ist solution : As $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$ is differentiable at $x = 3$, it must be first continuous at $x = 3$.

Hence, $\lim\limits_{x\to 3^+} g(x) = g(3) \Rightarrow 3m+2 = 2k$

Again, $g'^{+}(3) = \lim\limits_{h\to 0}\frac{g(3+h)-g(3)}{h}$

$$= \lim_{h\to 0}\frac{m(3+h)+2-2k}{h}$$

$$= \lim_{h\to 0}\frac{mh}{h} = m \quad (\text{as } 3m+2=2k)$$

Also, $g'^{-}(3) = \lim\limits_{\substack{h\to 0\\(h>0)}}\frac{g(3-h)-g(3)}{-h}$

$$= \lim_{h\to 0}\frac{k\sqrt{4-h}-2k}{-h} = \lim_{h\to 0}\frac{k(\sqrt{4-h}-2)}{-h}$$

$$= \lim_{h\to 0}\frac{k(4-h)-4}{-h}\cdot\frac{1}{\sqrt{4-h}+2} = \lim_{h\to 0}\frac{k}{\sqrt{4-h}+2} = \frac{k}{4}$$

Hence set $m = k/4$
Now, $3m + 2 = 2k$ yields $m = 2/5$, $k = 8/5$
$\therefore \; k + m = 2$

2nd solution : Since $g(x)$ is differentiable

$$\Rightarrow g'(x)|_{x=3} = \left.\frac{k}{2\sqrt{x+1}}\right|_{x=3} = \frac{k}{4} = m$$

(we earlier did it by 'ab initio' - first principles)
2nd solution can then be completed as before.
[Rating : Difficult]

25. (b) : Given, $\sum x_i = 256$
Now sum of all observations (after revision)
$= (256 - 16) + (3 + 4 + 5) = 240 + 12 = 252$
Also, the number of observations now become 18.
Hence, new mean $= \frac{252}{18} = 14$ **[Rating : Easy]**

26. (a) : Let $I = \int_2^4 \frac{\ln x^2}{\ln x^2 + \ln(x-6)^2}$

$$= \int_2^4 \frac{\ln x}{\ln x + \ln(6-x)} \quad \text{(use } \ln x^2 = 2\ln x,\ x > 0)$$

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, the above rewrite as

$$I = \int_2^4 \frac{\ln(6-x)}{\ln x + \ln(6-x)}$$

Adding the two, $2I = \int_2^4 \frac{\ln x + \ln(6-x)}{\ln x + \ln(6-x)} dx = \int_2^4 dx = 2$

$\therefore \quad I = 1.$

[Rating : Medium]

27. (a) : α is a root of $x^2 - 6x - 2 = 0$

Then $\alpha^2 - 6\alpha - 2 = 0$

Multiplying by α^n it become,

$$\alpha^{n+2} - 6\alpha^{n+1} - 2\alpha^n = 0$$

Similarly, $\beta^{n+2} - 6\beta^{n+1} - 2\beta^n = 0$

Subtracting, we get

$(\alpha^{n+2} - \beta^{n+2}) - 6(\alpha^{n+1} - \beta^{n+1}) - 2(\alpha^n - \beta^n) = 0$

i.e., $a_{n+2} - 6a_{n+1} - 2a_n = 0$

Thus, $\frac{a_{n+2} - 2a_n}{2a_{n+1}} = 3$

Set $n = 8$ to obtain the desired value $\frac{a_{10} - 2a_8}{2a_9} = 3$

[Rating : Difficult]

28. (a) : 1st solution :

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + \lambda$

As $\lim_{x\to 0} \left(1 + \frac{ax^4 + bx^3 + cx^2 + dx + \lambda}{x^2}\right) = 3$

We have $d = \lambda = 0$, the coefficient of exponents lower than 2 must vanish.

$\Rightarrow \quad \lim_{x\to 0} (1 + ax^2 + bx + c) = 3$

$\Rightarrow \quad c = 2$

$f(x) = ax^4 + bx^3 + 2x^2$

$f'(x) = 4ax^3 + 3bx^2 + 4x = x(4ax^2 + 3bx + 4)$

$x = 1$ and 2 are roots of $4ax^2 + 3bx + 4 = 0$

Thus, $-\frac{3b}{4a} = 3$ and $2 = \frac{4}{4a}$

(Using sum and product of roots)

Solving, we get $a = \frac{1}{2}, b = -2$

$$f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$$

Put 2 to get $f(2) = 8 - 16 + 8 = 0$.

2nd solution : As f has extreme values at $x = 1$ and $x = 2$, we build f from f'.

$f'(x) = k(x-1)(x-2)(x-\alpha)$

As f' is a polynomial of degree 3.

As $\lim\limits_{x\to 0}\left(1+\dfrac{f(x)}{x^2}\right)=3 \Rightarrow \lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=2$

Thus, $f(x) = x^2 g(x)$

Hence 0 is a repeated root of $f(x)$.

Here $f'(x) = k(x-1)(x-2)x = k(x^3 - 3x^2 + 2x)$

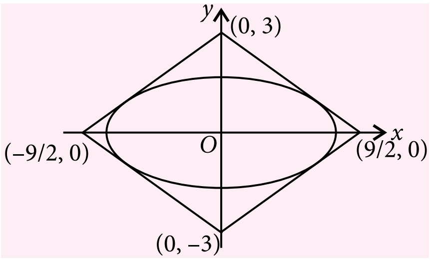
$$\Rightarrow f(x)=k\left(\frac{x^4}{4}-x^3+x^2\right)=kx^2\left(\frac{x^2}{4}-x+1\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2}=k=2$$

Thus, $f(x)=\dfrac{x^4}{2}-2x^3+2x^2$

[Rating : Difficult]

29. (b) : The ellipse is as described.



$$\frac{x^2}{9}+\frac{y^2}{5}=1 \text{ gives } a=3,\ b=\sqrt{5}$$

Using $b^2 = a^2(1 - e^2)$, we have

$$1-e^2=\frac{5}{9} \Rightarrow e^2=\frac{4}{9}$$

$\Rightarrow e = 2/3$

Equation of tangent at (2, 5/3) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \ \ i.e., \ \frac{x\cdot 2}{9}+\frac{y}{5}\cdot\left(\frac{5}{3}\right)=1$$

$$\therefore \ \frac{2x}{9}+\frac{y}{3}=1$$

The points of intersection with axes are (0, 3) and (9/2, 0) of the above line.

Using symmetry, picture can be completed.

Area required $=\dfrac{1}{2}(3)(9/2)\times 4=27$

[Rating : Medium]

30. ()   : There is an ambiguity in the system. If it is interpreted as that a particular box contains exactly 3 balls, then $n(E) = {}^{12}C_3 \cdot 2^9$

Probability $=\dfrac{n(E)}{n(S)}=\dfrac{{}^{12}C_3\cdot 2^9}{3^{12}}=\dfrac{55}{3}\left(\dfrac{2}{3}\right)^{11}$

But the system doesn't say that we have,

$n(E) = {}^{12}C_3 \cdot {}^3C_1 \times (2^9 - {}^9C_3 \cdot 2) + {}^{12}C_6 \cdot {}^3C_1 \times {}^6C_3$

Because one of the boxes contains exactly 3 balls, we have to subtract to make the count correct.

$n(E) = 57 \times 107 \times 24$

Probability $=\dfrac{n(E)}{n(S)}=\dfrac{57\times 107\times 24}{3^{11}}$

[Rating : Medium]

Singapore high school maths problem stumps the Internet

A maths problem that first appeared in a test for Singapore's elite high school students has baffled Internet users around the world after it went viral, prompting a rush of attempts to solve it.

The question, involving a girl asking two boys to guess her birthday after giving them scant clues, first appeared in an April 8 test organised by the Singapore and Asian School Math Olympiads (SASMO).

It was meant for 15- and 16-year-old elite secondary school students, but swiftly went global after a local television news presenter posted it on his Facebook page.

By Monday Internet users around the world were posting meticulously detailed answers to the puzzle on social media networks only to prompt a slew of comments disputing their findings and methodology.

Others posted sardonic comments about "Coy Cheryl".

The hashtag #cherylsbirthday also trended on Twitter. The correct answer to the question is July 16.

Singapore is renowned worldwide for its national maths system, which has been emulated by schools in other developed countries and cities like New York.

GIVE THE MATH QUESTION A GO

Albert and Bernard just became friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of 10 possible dates.

May 15, May 16, May 19, June 17, June 18, July 14, July 16, August 14, August 15, August 17.

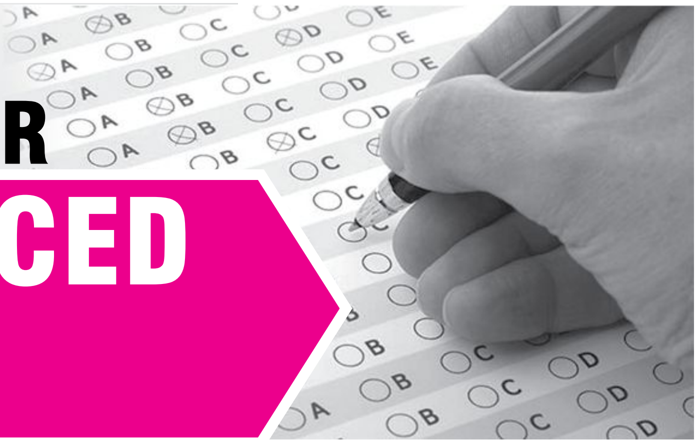
Cheryl then tells Albert and Bernard separately the month and the day of her birthday respectively.

Albert : I don't know when Cheryl's birthday is , but I know that Bernard does not know too.

Bernard : At first I don't know when Cheryl's birthday is, but I know now.

Albert : Then I also know when Cheryl's birthday is.

So when is Cheryl's birthday?



PRACTICE PAPER
JEE ADVANCED 2015

Exam on 24th May

SECTION-I

Only One Option Correct Type

This section contains 10 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. Number of six-digit number in which no digit is repeated, which is divisible by 5, odd numbers occur at even place and even numbers occur at odd place, is

(a) 400 (b) 625 (c) 225 (d) 576

2. The value of $\frac{3+6}{4^{100}}+\frac{3+2\times 6}{4^{99}}+\frac{3+3\times 6}{4^{98}}+\ldots+\frac{3+100\times 6}{4}$ is equal to

(a) $\frac{1}{3}\left(601-\left(\frac{1}{4}\right)^{100}\right)$ (b) $\frac{1}{3}\left(600-\left(\frac{1}{4}\right)^{100}\right)$

(c) $\frac{1}{3}\left(700-\left(\frac{1}{4}\right)^{99}\right)$ (d) $\frac{1}{3}\left(700-\left(\frac{1}{4}\right)^{100}\right)$

3. The family of lines

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y - (5\cos\theta - 2\sin\theta) = 0$$

always passes through the point (α, β) $\forall$ $\theta \in R$ then $\alpha + \beta$ is equal to

(a) 1 (b) -1 (c) 2 (d) 0

4. The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 4z - 5 = 0$. The equation of plane in its new position is

(a) $x + 3y - 5z = 17$ (b) $11x + 3y - z = 10$

(c) $3x + 4y - 11z = 12$ (d) $22x + 5y - 4z = 35$

5. $\int \frac{3x^2+4x-1}{(x^2+1)^2\sqrt{x+1}}dx$ is equal to

(a) $\frac{\sqrt{x+1}}{x^2+1}+c$ (b) $-\frac{2\sqrt{x+1}}{x^2+1}+c$

(c) $\frac{2}{(x^2+1)\sqrt{x+1}}+c$ (d) $-\frac{x}{(x^2+1)\sqrt{x+1}}+c$

6. The general solution of the differential equation $y\{xy - (x^2 - y^2)^2\}dx = \{y^3 - x(x^2 - y^2)^2\}dy$ is equal to

(a) $\frac{y}{x}+\frac{1}{x^2-y^2}=c$ (b) $\frac{x}{y}-\frac{1}{2}\ln\left|\frac{x-y}{x+y}\right|=c$

(c) $\frac{2x}{y}+\frac{1}{x^2-y^2}=c$ (d) $\frac{2x}{y}-\frac{1}{x^2-y^2}=c$

7. A line is drawn at an angle θ with positive direction of x-axis. It intersects the parabola $y^2 = 8x$ at A and B. If AB is a normal chord which subtend a right angle at the vertex of the parabola, then $\tan^2\theta$ equals

(a) 1/2 (b) 3/4 (c) 1 (d) 2

8. If $f(x)=\frac{3^{x-3}}{3^{1-x}+3^x}$ for all $x \in Q$, then the value of

$$f\left(\frac{1}{109}\right)+f\left(\frac{2}{109}\right)+f\left(\frac{3}{109}\right)+\ldots+f\left(\frac{108}{109}\right)$$ is

(a) 54 (b) 55 (c) 108 (d) 2

9. The value of $\cot \sum_{n=1}^{\infty} \cot^{-1}\left(\frac{2}{n^2}\sum_{k=1}^{n}k^3\right)$ is

(a) 1/2 (b) 1/4 (c) 1/3 (d) 1/6

10. The graph of the function $y = f(x)$ has a unique tangent at $(\pi^3, 0)$. If $A=\lim_{x\to\pi^3}\frac{\ln(1+9f(x))-\sin f(x)}{2f(x)}$, then $\sum_{n=1}^{\infty} A^{-n}$ is equal to

(a) 1/3 (b) 1/2 (c) 4 (d) 1/4

SECTION-II

More than One Option Correct Type

This section contains 5 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE are correct.

11. If the intercept made by circle $x^2 + y^2 + x + 5y + 4 = 0$ on lines $x + y + 1 = 0$ and $y = m(x + 1) - 1$ be equal then the values of m can be

(a) 1 (b) -1 (c) $-1/7$ (d) 7

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12. If (0, 0) is one of the vertices and $x - y - 2 = 0$ is one of the sides of an equilateral triangle, then

(a) area of triangle is $\frac{2}{\sqrt{3}}$ sq. unit

(b) orthocentre is $\left(\frac{2}{3}, -\frac{2}{3}\right)$

(c) slope of other two sides of triangle are $-(2+\sqrt{3})$ and $(\sqrt{3}-2)$

(d) circumcentre is $\left(\frac{1}{3}, -\frac{1}{3}\right)$

13. A normal, drawn at a point $P(x, y)$ of a curve meets the x-axis at A and y-axis at B such that $\frac{2}{OA}+\frac{3}{OB}=\frac{1}{5}$, where O is origin. If the curve passes through (7, 11) then

(a) the differential equation of the curve is $(x - 10)dx + (y - 15)dy = 0$

(b) the differential equation of the curve is $(x - 10)dx - (y - 15)dy = 0$

(c) curve passes through (13, 11)

(d) curve passes through (7, 19)

14. Let z_1 and z_2 be the roots of the equations

$$z^2 - 2z\cos\frac{2\pi}{7} + 1 = 0 \text{ and}$$

$$z^2 - 2z\cos\frac{3\pi}{7} + 1 = 0 \quad \text{respectively.}$$

Then which of the following is/are true?

(a) $z_1^2 z_2 + \frac{1}{z_1^2 z_2} = 2$ (b) $z_1^2 z_2 + \frac{1}{z_1^2 z_2} = -2$

(c) $\frac{z_1^3}{z_2^2} + \frac{z_2^2}{z_1^3} = 2$ (d) $\frac{z_1^3}{z_2^2} + \frac{z_2^2}{z_1^3} = -2\cos\frac{5\pi}{7}$

15. The function $f(x)$ satisfying $(f(x))^2 - 4f(x)\,f'(x) + (f'(x))^2 = 0$ is given by

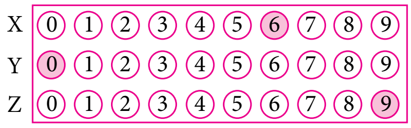
(a) $ke^{(2+\sqrt{3})x}$ (b) $ke^{(2-\sqrt{3})x}$

(c) $ke^{(2+\sqrt{6})x}$ (d) $ke^{(2-\sqrt{6})x}$

SECTION-III

Integer Value Correct Type

This section contains 5 questions. The answer to each of the question is a single digit integer, ranging from 0 t0 9. The appropriate bubbles corresponding the respective question numbers in the ORS have to be darkened l. For example, if the correct answers to question numbers X, Y and Z (say) are 6, 0 and 9, respectively, then the correct darkening of bubbles will look like the following :



16. The maximum value of the function

$$g(x) = \frac{106}{3x^4 + 8x^3 - 18x^2 + 60}, (x > 0) \text{ is } \_\_\_\_\_\_.$$

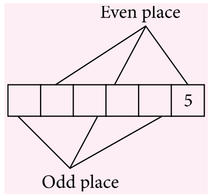
17. Let $p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6$ for k with $0 < k < 5$, define $I_k = \int_0^\infty \frac{x^k}{p(x)}dx$, the value of k for which I_k is smallest is ______.

18. The number of points where $f(x) = |2 - |2 - |x|||$ is not differentiable is ______.

19. If the sum of first n terms of the A.P. 85, 90, 95, ... is equal to the sum of first $3n$ terms of the A.P. 9, 11, 13, ..., then n equals ______.

20. A nine digit number is formed from 1, 2, 3, 4, 5 such that product of all digits is always 1920. The total number of ways is $393({}^9P_r)$, where the value of r is

SOLUTIONS



1. (d) : Even digits = 0, 2, 4, 6, 8

Odd digits = 1, 3, 5, 7, 9

Required no = 4 × 3 × 4 × 4 × 3

= 576

2. (a) :

$$\underbrace{3\left(\frac{1}{4}+\frac{1}{4^2}+....+\frac{1}{4^{100}}\right)}_{S_1}, +6\underbrace{\left(\frac{1}{4^{100}}+\frac{2}{4^{99}}+....+\frac{100}{4}\right)}_{S_2}$$

$$\text{Now, } S_1 = \frac{3}{4}\left(\frac{1-\left(\frac{1}{4}\right)^{100}}{1-\frac{1}{4}}\right) = 1-\left(\frac{1}{4}\right)^{100}$$

$$\text{and } S_2 = \frac{1}{4^{100}}+\frac{2}{4^{99}}+....+\frac{100}{4}$$

$$\text{or } 4S_2 = \frac{1}{4^{99}}+.....+100$$

$$-3S_2 = \frac{1}{4^{100}}+\left(\frac{1}{4^{99}}+\frac{1}{4^{98}}+....+\frac{1}{4}\right)-100$$

$$\Rightarrow S_2 = \frac{100}{3}-\frac{1}{9}\left(1-\left(\frac{1}{4}\right)^{100}\right)$$

$$\therefore S = \left(1-\left(\frac{1}{4}\right)^{100}\right)+6\left(\frac{100}{3}-\frac{1}{9}\left(1-\left(\frac{1}{4}\right)^{100}\right)\right)$$

$$=\frac{1}{3}\left(601-\left(\frac{1}{4}\right)^{100}\right)$$

3. (c) : $\cos\theta(2x + 3y - 5) + \sin\theta\,(3x - 5y + 2) = 0$

Required point is intersecting point of

$2x + 3y - 5 = 0$ and $3x - 5y + 2 = 0$

$\Rightarrow \alpha = 1, \beta = 1 \quad \therefore \alpha + \beta = 2$

4. (d) : Equation of plane through the line of intersection of $x - 2y + 3z = 0$ and $2x + 3y - 4z - 5 = 0$ is

$x - 2y + 3z + \lambda(2x + 3y - 4z - 5) = 0$ ....(i)

(i) is perpendicular to $x - 2y + 3z = 0$

$\therefore\ 1(1 + 2\lambda) - 2(3\lambda - 2) + 3(3 - 4\lambda) = 0$

$\Rightarrow\ \lambda = 7/8$

Required equation of plane is $22x + 5y - 4z = 35$

5. (b) : $I = \int \frac{3x^2+4x-1}{(x^2+1)^2\sqrt{x+1}}dx$

$$= \int \frac{4x^2+4x-x^2-1}{(x^2+1)^2\sqrt{x+1}}dx$$

$$= 4\int \frac{x(x+1)}{(x^2+1)^2\sqrt{x+1}}dx - \int \frac{dx}{(x^2+1)(\sqrt{x+1})}dx$$

$$= 4\int \frac{x(x+1)}{(x^2+1)^2\sqrt{x+1}}dx - \left\{\frac{1}{(x^2+1)}2\sqrt{x+1} - \int \frac{-2x}{(x^2+1)^2}2\sqrt{x+1}\,dx\right\}$$

$$= -\frac{2\sqrt{x+1}}{x^2+1} + c$$

6. (c) : $xy^2dx - y(x^2 - y^2)^2dx = y^3dy - x(x^2 - y^2)^2dy$

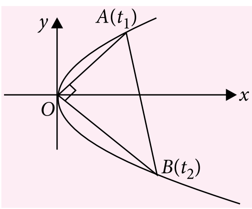
$\Rightarrow\ y^2(xdx - ydy) = (x^2 - y^2)^2(ydx - xdy)$

$$\Rightarrow\ \frac{xdx-ydy}{(x^2-y^2)^2} = \frac{ydx-xdy}{y^2}\ \Rightarrow\ \frac{d(x^2-y^2)}{(x^2-y^2)^2} = 2d\left(\frac{x}{y}\right)$$

Integrating, we get

$$\frac{2x}{y} + \frac{1}{(x^2-y^2)} = c$$

7. (d) : AB is a normal chord



$\therefore\ t_2 = -t_1 - \frac{2}{t_1}$ ...(i)

Also, AB subtends right angle at the vertex

$\therefore\ t_1t_2 = -4\ \Rightarrow\ t_2 = \frac{-4}{t_1}$ ...(ii)

From (i) and (ii), we have

$-\frac{4}{t_1} = -t_1 - \frac{2}{t_1}\ \Rightarrow\ t_1^2 = 2$ ...(iii)

Equation of AB is $y + t_1x = 2at_1 + at_1^3$

$\therefore$ The slope of $AB = -t_1 = \tan\theta$

$\Rightarrow\ \tan^2\theta = t_1^2 = 2$ (from (iii))

8. (d) : $f(x) + f(1-x) = \frac{1}{27}$

$\Rightarrow\ \frac{1}{27} \times 54 = 2$

9. (c) : $\cot \sum_{n=1}^{\infty} \cot^{-1}\frac{1}{n^2} \times \frac{n^2(n+1)^2}{2}$

$$= \cot \sum_{n=1}^{\infty} \tan^{-1}\frac{2}{(n+1)^2} = \cot \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{n+2-n}{1+n(n+2)}\right)$$

$$= \cot \underbrace{\sum_{n=1}^{\infty} \tan^{-1}(n+2) - \tan^{-1}n}_{S_n}$$

$$S_n = \begin{bmatrix} \tan^{-1}3 - \tan^{-1}1 \\ \tan^{-1}4 - \tan^{-1}2 \\ \tan^{-1}5 - \tan^{-1}3 \\ \vdots \\ \tan^{-1}(n+2) - \tan^{-1}(n) \end{bmatrix}$$

$S_n = \tan^{-1}(n + 2) + \tan^{-1}(n + 1) - \tan^{-1} 2 - \tan^{-1}1$

$S_n = \tan^{-1}\left(\frac{3n^2+7n}{n^2+9n+10}\right),\ S_\infty = \tan^{-1}3$

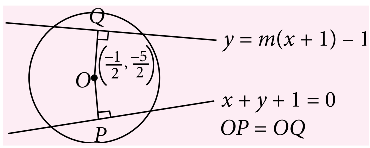
Then $\cot \tan^{-1} 3 = \cot\ \cot^{-1}\left(\frac{1}{3}\right) = \frac{1}{3}$

10 (a) : $$\lim_{x\to\pi^3} \frac{\left\{\frac{1}{1+9f(x)}\right\}9f'(x) - \cos f(x)\cdot f'(x)}{2f'(x)} = \frac{9-1}{2} = 4$$

$$\sum_{n=1}^{\infty} 4^{-n} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + ...\infty\ \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

11. (a, c) : $$\left|\frac{-\frac{1}{2} - \frac{5}{2} + 1}{\sqrt{2}}\right| = \left|\frac{\frac{3}{2} + \frac{m}{2}}{\sqrt{1+m^2}}\right|$$

$$\Rightarrow\ 2 = \frac{\frac{9}{4} + \frac{m^2}{4} + \frac{3m}{2}}{(1+m^2)}$$



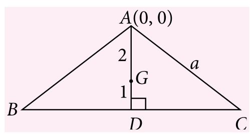
$\Rightarrow\ 2 + 2m^2 = \frac{9}{4} + \frac{m^2}{4} + \frac{3m}{2}\ \Rightarrow\ 7m^2 - 6m - 1 = 0$

$\Rightarrow\ 7m^2 - 7m + m - 1 = 0\ \Rightarrow\ 7m(m - 1) + 1(m - 1) = 0$

$\Rightarrow\ m = 1,\ -\frac{1}{7}$

12. (a, b, c) : Let equation of BC is $x - y - 2 = 0$, Slope of line $x - y - 2 = 0$ is $\tan\theta = 1$

$AD = \left|\frac{2}{\sqrt{2}}\right| = \sqrt{2} = \frac{\sqrt{3}}{2}a$



$\therefore\ a = \frac{2\sqrt{2}}{\sqrt{3}}$

Area $= \frac{\sqrt{3}}{4} \times \frac{8}{3} = \frac{2}{\sqrt{3}}$ sq. unit

Equation of AD is $x + y = 0$
$D = (1, -1)$

$\therefore \quad G = \left(\frac{2}{3}, -\frac{2}{3}\right) = \text{orthocenter} = \text{circumcenter}$

Slopes of other lines are $\tan(\theta \pm 60) = \dfrac{1 \pm \sqrt{3}}{1 \mp \sqrt{3}}$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}, \frac{1-\sqrt{3}}{1+\sqrt{3}} = -(2+\sqrt{3}), (\sqrt{3}-2)$$

13. (a, c, d) : The equation of normal at P is

$$Y - y = -\frac{1}{\left(\frac{dy}{dx}\right)}(X - x)$$

$\Rightarrow \quad OA = x + yy'$ and $OB = \dfrac{x}{y'} + y$

$$\therefore \quad \frac{2}{OA} + \frac{3}{OB} = \frac{1}{5}$$

$$\Rightarrow \quad \frac{2}{x+yy'} + \frac{3y'}{x+yy'} = \frac{1}{5}$$

$\Rightarrow \quad 10 + 15y' = x + yy' \Rightarrow (y - 15)dy + (x - 10)dx = 0$
$\Rightarrow \quad (x - 10)^2 + (y - 15)^2 = c$, it passes through (7, 11)
$\Rightarrow \quad c = 25 \qquad \therefore \quad (x - 10)^2 + (y - 15)^2 = 25$
It also passes through (13, 11) and (7, 19).

14. (b, c, d) : $z_1 = \cos\frac{2\pi}{7} \pm i\sin\frac{2\pi}{7} = e^{\frac{2\pi}{7}i}$ or $e^{-\frac{2\pi}{7}i}$

$$z_2 = \cos\frac{3\pi}{7} \pm i\sin\frac{3\pi}{7} = e^{\frac{3\pi}{7}i} \text{ or } e^{\frac{-3\pi}{7}i}$$

$$\therefore \quad z_1^2 z_2 + \frac{1}{z_1^2 z_2} = e^{\frac{4\pi}{7}i} \cdot e^{\frac{3\pi}{7}i} + \frac{1}{e^{\frac{4\pi}{7}i} e^{\frac{3\pi}{7}i}} = -2$$

$$\frac{z_1^3}{z_2^2} + \frac{z_2^2}{z_1^3} = \frac{e^{\frac{6\pi}{7}i}}{e^{\frac{6\pi}{7}i}} + \frac{e^{\frac{6\pi}{7}i}}{e^{\frac{6\pi}{7}i}} = 2$$

(if z_1 and z_2 are $e^{\frac{2\pi}{7}i}$ and $e^{\frac{3\pi}{7}i}$ respectively)

$$\text{Also } \frac{z_1^3}{z_2^2} + \frac{z_2^2}{z_1^3} = \frac{e^{\frac{6\pi}{7}i}}{e^{\frac{-6\pi}{7}i}} + \frac{e^{\frac{-6\pi}{7}i}}{e^{\frac{6\pi}{7}i}}$$

$$= 2\cos\frac{12\pi}{7} = -2\cos\frac{5\pi}{7}$$

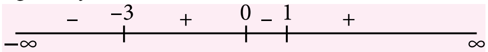
15. (a, b) : $\{f(x)\}^2 - 4f(x) \cdot f'(x) + \{f'(x)\}^2 = 0$

$$\Rightarrow \quad f'(x) = \frac{4f(x) \pm \sqrt{16(f(x))^2 - 4(f(x))^2}}{2}$$

$$\therefore \quad \frac{f'(x)}{f(x)} = 2 \pm \sqrt{3} \quad \therefore \quad \log_e f(x) = (2 \pm \sqrt{3})x + c$$

$\Rightarrow f(x) = k\,e^{(2\pm\sqrt{3})x}$

16. (2) : Let $f(x) = 3x^4 + 8x^3 - 18x^2 + 60$
$f'(x) = 12x^3 + 24x^2 - 36x$
$= 12(x^3 + 2x^2 - 3x) = 12x(x^2 + 2x - 3)$
$= 12x(x + 3)(x - 1)$
Also sign of $f'(x)$



$x \neq -3$ and $x = 1$ is point of local minima.
Now, $f(1) = 3 + 8 - 18 + 60 = 53$

$$\therefore \quad g(x)_{\max} = \frac{106}{53} = 2$$

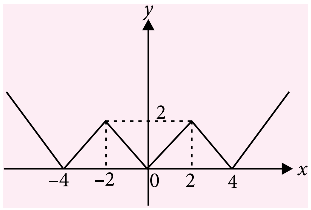
17. (2) : $l_k = \int_{\infty}^{0} \dfrac{t^{-k}}{t^{-6}p(t)}\left(\dfrac{-dt}{t^2}\right) = l_{4-k}$

$$l_k = \frac{(l_k + l_{4-k})}{2} = \int_0^{\infty} \frac{(x^k + x^{4-k})}{2p(x)}dx \geq \int_0^{\infty} \frac{x^2 dx}{p(x)} = l_2$$

$$\left(\frac{x^k + x^{4-k}}{2} \geq x^2 \text{ (AM-GM inequality)}\right)$$

l_k is smallest for $k = 2$.

18. (5) : As shown in the figure, $f(x)$ is non differentiable at 5 points.



19. (9) : $\dfrac{n}{2}[2\times85 + (n-1)\times5] = \dfrac{3n}{2}[2\times9 + (3n-1)\times2]$

$\Rightarrow \quad 5n + 165 = 48 + 18n$
$\Rightarrow \quad 13n = 117 \quad \Rightarrow \quad n = 9$

20. (2) : $1920 = 5 \times 3 \times 2^7$
$\Rightarrow \quad 1920 = 5 \times 3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$$= \frac{9!}{7!} = 9 \times 8$$

$\Rightarrow \quad 1920 = 5 \times 3 \times 4 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1$

$$= \frac{9!}{5!} = 9 \times 8 \times 7 \times 6$$

3, 5, 4, 4, 2, 2, 2, 1, 1

$$\Rightarrow \quad \frac{9!}{2!\,2!\,3!}$$

3, 5, 4, 4, 4, 2, 1, 1, 1

$$\Rightarrow \quad \frac{9!}{3!\,3!}$$

$$\therefore \quad \text{Total} = \frac{9!}{7!} + \frac{9!}{5!} + \frac{9!}{2!\,2!\,3!} + \frac{9!}{3!\,3!}$$

$$= \frac{9!}{7!}[1 + 42 + 210 + 140] = 393 \cdot {}^9P_2$$